CS245 Logic and Computation

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Contents

1 Propositional Logic

1.1 Translations

Exercise 1. *Translate the following three sentences into propositional logic.*

- **Nadhi will eat a fruit if it is an apple.**
- **Nadhi will eat a fruit only if it is an apple.**
- **Nadhi will eat a fruit if and only if it is an apple.**

n: Nadhi will eat a fruit.
a: The fruit is an apple.

- **Nadhi will eat a fruit if it is an apple.**

  Translation: (a → n)

  If the fruit is an apple, we know that Nadhi will eat it.

  If the fruit is not an apple, Nadhi may or may not eat it.

  The set of apples is a subset of the set of fruits that Nadhi eats.

- **Nadhi will eat a fruit only if it is an apple.**

  Translation: (n → a)

  If Nadhi eats a fruit, then we know that it is an apple.

  If Nadhi does not eat a fruit, the fruit may or may not be an apple.

  The set of fruits that Nadhi eats is a subset of the set of apples.

- **Nadhi will eat a fruit if and only if it is an apple.**

  Translation: (n ↔ a)

  If Nadhi eats a fruit, then it is an apple.

  If Nadhi does not eat a fruit, then it is not an apple.

  The set of fruits that Nadhi eats and the set of apples coincide.

Exercise 2. *Translate the following sentence into multiple propositional formulas. Show that they are logically equivalent using a truth table.*

Soo-Jin will eat an apple or an orange but not both.

a: Soo-Jin will eat an apple. o: Soo-Jin will eat an orange.

This sentence translates into an exclusive OR. There are many ways of writing down a formula for an exclusive OR.

- $((a \lor o) \land (\neg(a \land o)))$

  a or o is true, but not both.

- $((a \lor o) \land ((\neg a) \lor (\neg o)))$

  a or o is true, and a is false or o is false.

- $((a \land (\neg o)) \lor ((\neg a) \land o))$

  a is true and o is false, or a is false and o is true.

- $(\neg(a \leftrightarrow o))$

  It is not the case that a and o have the same truth value.

- $((\neg a) \leftrightarrow o) \equiv (a \leftrightarrow (\neg o))$

  negated a and o have the same truth value.

Exercise 3. *Translate the following sentence into at least three syntactically different propositional formulas. Show that they are logically equivalent using a truth table.*

If it is sunny tomorrow, then I will play golf, provided that I am relaxed.

- s: It is sunny tomorrow.
- g: I will play golf.
- r: I am relaxed.

I can think of three ways of translating this sentence into a propositional formula.

- Interpretation 1: If it is sunny tomorrow, then, if I am relaxed, then I will play golf.

  Translation: $(\mathrm{s} \rightarrow (\mathrm{r} \rightarrow \mathrm{g}))$.

  Sunny tomorrow is the premise for the first.

- Interpretation 2: If it is sunny tomorrow and I am relaxed, then I will play golf.

  Translation: $((\mathrm{s} \wedge \mathrm{r}) \rightarrow \mathrm{g})$.

  Sunny tomorrow and being relaxed together are premises for playing golf.

- Interpretation 3: If I am relaxed, then, if it is sunny tomorrow, I will play golf.

  Translation: $(\mathrm{r} \rightarrow (\mathrm{s} \rightarrow \mathrm{g}))$.

  Being relaxed is the premise for the rest.

All three interpretations are logically equivalent.

Exercise 4. *Translate the following sentence into a propositional formula.*

If I ace CS 245, I will get a job at Google; otherwise I will apply for the Geek Squad.

Define the propositional variables:

- a: I ace CS 245.
- g: I will get a job at Google.
- s: I will apply for the Geek Squad.

First, let's break down this sentence into two parts by the semicolon.

The first part translates into an implication because of the key word "if". It becomes $(a \to g)$.

In the second part, "otherwise" means that "if I don't ace CS 245". After rephrasing, the second part becomes "If I don't ace CS 245, then I will apply for the Geek Squad." This is another implication $((\neg a) \to s)$.

Now the tricky part is: what connective should we use to connect the two parts together? Two natural options are $\land$ and $\lor$. The $\lor$ option seems possible because the sentence could be rephrase as "If I ace CS 245, ...; or otherwise ...."
The correct connective to use is $\land$ for the following reasons.

Let's consider the scenario in which I ace CS 245, I don't get a job at Google and I apply for the Geek Squad. In this case, is the sentence true or false? Intuitively, the sentence should be false, because the first implication is violated when I ace CS 245 but do not get a job at Google. Now let's look at the truth values of the two possible propositional formulas:

- If we use $\land$ as the connective, the resulting formula $((a \to g) \land ((\neg a) \to s))$ is false in this scenario. The truth value of the formula is the same as the truth value of the sentence in this scenario.
- If we use $\lor$ as the connective, the resulting formula $((a \to g) \land ((\neg a) \to s))$ is true in this scenario. This truth value of the formula is different from the truth value of the sentence in this scenario. Therefore, $\lor$ is not the correct connective to use because the resulting formula has a different meaning from the formula.

Exercise 5. *Translate the following sentence into two propositional formulas and prove that the formulas are not logically equivalent using a truth table.*

Sidney will carry an umbrella unless it is sunny.

Define the propositional variables.
u: Sidney will carry an umbrella.
s: It is sunny.

- Interpretation 1:

  Intuitively, many people understand "unless" as an "exclusive OR", which means that exactly one of the two parts of the sentence is true at a time.

  With this interpretation, "unless" is equivalent to an "if and only if not". The sentence is true under the following two scenarios:

  - It is not sunny and Sidney carries an umbrella.
  - It is sunny and Sidney does not carry an umbrella.

  Note that this interpretation does not allow Sidney to carry an umbrella when it is sunny. So the sentence is false when u and s are both true.

  In propositional logic, this is equivalent to

$$
\begin{aligned}
& ((\neg \mathrm{u}) \leftrightarrow \mathrm{s}) && (1)\\
\equiv\ & (((\neg \mathrm{u}) \land \mathrm{s}) \lor (\mathrm{u} \land (\neg \mathrm{s}))) && (2)\\
\equiv\ & ((\mathrm{u} \lor \mathrm{s}) \land (\neg(\mathrm{u} \land \mathrm{s}))) && (3)\\
\equiv\ & ((\mathrm{u} \lor \mathrm{s}) \land ((\neg \mathrm{u}) \lor (\neg \mathrm{s}))). && (4)
\end{aligned}
$$

  All the formulas above are equivalent. They look different but their meanings are the same.

- Interpretation 2:

  Alternatively, you may think of "unless" as meaning "if not". Then the sentence becomes: if it is not sunny, then Sidney will carry an umbrella. In propositional logic, this becomes:

$$
\begin{aligned}
& ((\neg \mathrm{s}) \to \mathrm{u}) && (5)\\
\equiv\ & ((\neg(\neg \mathrm{s})) \lor \mathrm{u}) && (6)\\
\equiv\ & (\mathrm{s} \lor \mathrm{u}). && (7)
\end{aligned}
$$

  Under this interpretation, this sentence is true under three scenarios:

  - It is not sunny and Sidney carries an umbrella.

- It is sunny and Sidney does not carry an umbrella.
- It is sunny and Sidney carries an umbrella.

Notice that this interpretation allows Sidney to carry an umbrella when it is sunny. So the sentence is true when u and s are both true.

1.2 Structural Induction

Theorem 1. *Every well-formed formula has an equal number of opening and closing brackets.*

Proof by Structural Induction. Let $P(\varphi)$ denote that the well-formed formula φ has an equal number of opening and closing brackets. Let $op(\varphi)$ and $cl(\varphi)$ denote the number of opening and closing brackets of φ respectively.

Base case: φ is a propositional variable q. Prove that $P(q)$ holds.

q has zero opening and zero closing bracket. Thus, $P(\varphi)$ holds.

Induction step:

Case 1: φ is $(\neg a)$, where a is well-formed.
Induction hypothesis: Assume that $P(a)$ holds.
We need to prove that $P((\neg a))$ holds.

$$\begin{aligned} op((\neg a)) &= 1 + op(a) && (8) \\ &= 1 + cl(a) \text{ By induction hypothesis} && (9) \\ &= cl((\neg a)) && (10) \end{aligned}$$

Thus, $P((\neg a))$ holds.

Case 2: φ is $(a * b)$ where a and b are well-formed and $*$ is a binary connective.
Induction hypothesis: Assume that $P(a)$ and $P(b)$ hold.
We need to prove that $P((a * b))$ holds.

$$\begin{aligned} op((a * b)) &= 1 + op(a) + op(b) && (11) \\ &= 1 + cl(a) + cl(b) \text{ By induction hypothesis} && (12) \\ &= cl(a * b) && (13) \end{aligned}$$

Thus, $P((a * b))$ holds.

By the principle of structural induction, $P(\varphi)$ holds for every well-formed formula φ. QED

□

Theorem 2. *Every proper prefix of a well-formed formula has more opening than closing brackets.*

Proof by Structural Induction. Let $P(\varphi)$ denote that every proper prefix of the well-formed formula φ has more opening than closing brackets.
Let $op(\varphi)$ and $cl(\varphi)$ denote the number of opening and closing brackets of φ respectively.

Base case: φ is a propositional variable q. Prove that $P(q)$ holds.

Induction step:

Case 1: φ is $(\neg a)$, where a is well-formed.
Induction hypothesis: Assume that $P(a)$ holds.
We need to prove that $P((\neg a))$ holds.
Let m denote any proper prefix of a. There are four possible proper prefixes of $(\neg a)$: $($, $(\neg$, $(\neg m$, and $(\neg a$. We will prove the four cases separately.

$$\mathrm{op}(() = 1 \tag{14}$$
$$\mathrm{cl}(() = 0 \tag{15}$$
$$\mathrm{op}(() > \mathrm{cl}(() \tag{16}$$

$$\mathrm{op}((\neg) = 1 \tag{17}$$
$$\mathrm{cl}((\neg) = 0 \tag{18}$$
$$\mathrm{op}((\neg) > \mathrm{cl}(() \tag{19}$$

$$\mathrm{op}((\neg m) \tag{20}$$
$$= 1 + \mathrm{op}(m) \tag{21}$$
$$> 1 + \mathrm{cl}(m) \text{ By the induction hypothesis on m} \tag{22}$$
$$> \mathrm{cl}(m) \tag{23}$$
$$= \mathrm{cl}((\neg m) \tag{24}$$

$$\mathrm{op}((\neg a) \tag{25}$$
$$= 1 + \mathrm{op}(a) \tag{26}$$
$$= 1 + \mathrm{cl}(a) \text{ By Theorem 1 and a is a well-formed formula} \tag{27}$$
$$> \mathrm{cl}(a) \tag{28}$$
$$= \mathrm{cl}((\neg a) \tag{29}$$

Case 2: φ is $(a * b)$ where a and b are well-formed and $*$ is a binary connective. Let m and n denote any proper prefix of a and b respectively.

Induction hypothesis: Assume that P(a) and P(b) hold. In other words, P(m) and P(n) are true.

We need to prove that $P((a * b))$ holds.

There are six possible proper prefixes of $(a * b)$: (, (m, (a, (a∗, (a ∗ n, and (a ∗ b.

$$\mathrm{op}(() = 1 \tag{30}$$
$$\mathrm{cl}(() = 0 \tag{31}$$
$$\mathrm{op}(() > \mathrm{cl}(() \tag{32}$$

$$\mathrm{op}((m) \tag{33}$$
$$= 1 + \mathrm{op}(m) \tag{34}$$
$$> 1 + \mathrm{cl}(m) \text{ By the induction hypothesis on m} \tag{35}$$
$$> \mathrm{cl}(m) \tag{36}$$
$$= \mathrm{cl}((m) \tag{37}$$

$$\mathrm{op}((a) \tag{38}$$
$$= 1 + \mathrm{op}(a) \tag{39}$$
$$= 1 + \mathrm{cl}(a) \text{ By Theorem 1 and a is a well-formed formula} \tag{40}$$
$$> \mathrm{cl}(a) \tag{41}$$
$$= \mathrm{cl}((a) \tag{42}$$

$$\mathrm{op}((a*) \tag{43}$$
$$= 1 + \mathrm{op}(a) \tag{44}$$
$$= 1 + \mathrm{cl}(a) \text{ By Theorem 1 and a is a well-formed formula} \tag{45}$$
$$> \mathrm{cl}(a) \tag{46}$$
$$= \mathrm{cl}((a*) \tag{47}$$

$$\text{op}((\text{a} * \text{n}) \tag{48}$$
$$= 1 + \text{op}(\text{a}) + \text{op}(\text{n}) \tag{49}$$
$$= 1 + \text{cl}(\text{a}) + \text{op}(\text{n}) \text{ By Theorem 1 and a is a well-formed formula} \tag{50}$$
$$> 1 + \text{cl}(\text{a}) + \text{cl}(\text{n}) \text{ By the induction hypothesis on n} \tag{51}$$
$$> \text{cl}(\text{a}) + \text{cl}(\text{n}) \tag{52}$$
$$= \text{cl}((\text{a} * \text{n}) \tag{53}$$

$$\text{op}((\text{a} * \text{b}) \tag{54}$$
$$= 1 + \text{op}(\text{a}) + \text{op}(\text{b}) \tag{55}$$
$$= 1 + \text{cl}(\text{a}) + \text{cl}(\text{b}) \text{ By Theorem 1 and a is a well-formed formula} \tag{56}$$
$$> \text{cl}(\text{a}) + \text{cl}(\text{b}) \tag{57}$$
$$= \text{cl}((\text{a} * \text{b}) \tag{58}$$

By the principle of structural induction, P(φ) holds for every well-formed formula φ. QED

□

1.3 Tautology, Contradiction, and Satisfiable but Not a Tautology

Exercise 6. *Determine whether each of the following formulas is a tautology, satisfiable but not a tautology, or a contradiction.*

- p

  Answer: Satisfiable but not a tautology.

  Reason: True when p is true and false when p is false.

- $((r \wedge s) \rightarrow r)$

  Answer: Tautology.

  Reason: When r is true, the conclusion of the implication is true, so the implication is true. When r is false, the premise of the implication is false, so the implication is vacuously true.

- $((\neg(p \leftrightarrow q)) \leftrightarrow (q \vee p))$

  Answer: Satisfiable but not a tautology

  Reason: It's tempting to say "these two formulas don't mean the same thing so the biconditional is false". However, go back to truth values. When p is true and q is false, both sides of the biconditional are true and the biconditional itself is true. When p and q are both true, the left side is false but the right is true, and so the biconditional is false.

- $((((p \vee q) \wedge (p \vee (\neg q))) \wedge ((\neg p) \vee q)) \wedge ((\neg p) \vee (\neg q)))$

  Answer: Contradiction

  Reason: The first half can be simplfiied to $(p \vee (q \wedge (\neg q)))$, which is $(p \vee F)$ or p. The second half can be simplfiied to $(\neg p)$. Thus, the entire formula is $(p \wedge (\neg p))$, which is a contradiction.

1.4 Logical Equivalence

Exercise 7. *"If it is sunny, I will play golf, provided that I am relaxed."*
s*: it is sunny.* g*: I will play golf.* r*: I am relaxed.*

There are three possible translations:

1. $(r \to (s \to g))$
2. $((s \wedge r) \to g)$
3. $(s \to (r \to g))$

Prove that all three translations are logically equivalent.

Part 1: $(r \to (s \to g)) \equiv ((s \wedge r) \to g)$.

Proof.

$$\begin{aligned}
&(r \to (s \to g)) && && (59)\\
&\equiv (r \to ((\neg s) \vee g)) && \text{Implication} && (60)\\
&\equiv ((\neg r) \vee ((\neg s) \vee g)) && \text{Implication} && (61)\\
&\equiv (((\neg r) \vee (\neg s)) \vee g) && \text{Associativity} && (62)\\
&\equiv (((\neg (r \wedge s)) \vee g) && \text{De Morgan} && (63)\\
&\equiv ((r \wedge s) \to g) && \text{Implication} && (64)\\
&\equiv ((s \wedge r) \to g) && \text{Commutativity} && (65)
\end{aligned}$$

□

Part 2: $(r \to (s \to g)) \equiv (s \to (r \to g))$.

Proof.

$$\begin{aligned}
&(r \to (s \to g)) && && (66)\\
&\equiv (r \to ((\neg s) \vee g)) && \text{Implication} && (67)\\
&\equiv ((\neg r) \vee ((\neg s) \vee g)) && \text{Implication} && (68)\\
&\equiv (((\neg r) \vee (\neg s)) \vee g) && \text{Associativity} && (69)\\
&\equiv (((\neg s) \vee (\neg r)) \vee g) && \text{Commutativity} && (70)\\
&\equiv ((\neg s) \vee ((\neg r) \vee g)) && \text{Associativity} && (71)\\
&\equiv ((\neg s) \vee (r \to g)) && \text{Implication} && (72)\\
&\equiv (s \to (r \to g)) && \text{Implication} && (73)
\end{aligned}$$

□

Exercise 8. *"If it snows then I will not go to class but I will do my assignment."*
s*: it snows.* c*: I will go to class.* a*: I will do my assignment.*

There are two possible translations:

1. $((s \to (\neg c)) \wedge a)$
2. $(s \to ((\neg c) \wedge a))$

Prove that the two translations are NOT logically equivalent.

Proof. We need to find a valuation t under which the two formulas have different values.
Consider the truth valuation t where $t(s) = \mathtt{F}$, $t(c) = \mathtt{T}$, and $t(a) = \mathtt{F}$.
The two formulas have different values under t, as shown below.

- $((s \to (\neg c)) \wedge a)^t = \mathtt{F}$
- $(s \to ((\neg c) \wedge a))^t = \mathtt{T}$

□

1.5 Analyzing Conditional Code

Consider the following code fragment:

```
if (input > 0 || !output) {
  if (!(output && queuelength < 100)) {
    P1
  } else if (output && !(queuelength < 100)) {
    P2
  } else {
        P3
  }
} else {
  P4
}
```

Define the propositional variables:

- i: input > 0
- u: output
- q: queuelength < 100

The code fragment becomes the following. We'll call this code fragment #1.

```
if ( i || !u ) {
  if ( !(u && q) ) {
    P1
  } else if ( u && !q ) {
    P2
  } else { P3 }
} else { P4 }
```

Code fragment #2:

```
if (( i && u) && q) {
  P3
} else if (!i && u) {
  P4
} else {
  P1
}
```

Prove that these two pieces of code fragments are equivalent:

Prove that the condition leading to P_2 is logically equivalent to $\texttt{F}$.
The condition leading to P_2:

$$
\begin{aligned}
&(((i \vee (\neg u)) \wedge (\neg(\neg(u \wedge q)))) \wedge (u \wedge (\neg q))) && && (74)\\
&\equiv (((i \vee (\neg u)) \wedge (u \wedge q)) \wedge (u \wedge (\neg q))) && \text{Double Negation} && (75)\\
&\equiv ((i \vee (\neg u)) \wedge ((u \wedge u) \wedge (q \wedge (\neg q)))) && \text{Associativity, Commutativity} && (76)\\
&\equiv ((i \vee (\neg u)) \wedge (u \wedge (q \wedge (\neg q)))) && \text{Idempotence} && (77)\\
&\equiv ((i \vee (\neg u)) \wedge (u \wedge \texttt{F})) && \text{Contradiction} && (78)\\
&\equiv ((i \vee (\neg u)) \wedge \texttt{F}) && \text{Simplification 1} && (79)\\
&\equiv \texttt{F} && \text{Simplification 1} && (80)\\
& && && (81)
\end{aligned}
$$

Prove that the condition leading to P_3 is true if and only if all three variables are true.
The condition leading to P_3:

$$
\begin{aligned}
&(((i \vee (\neg u)) \wedge (u \wedge q)) \wedge (\neg(u \wedge (\neg q)))) && && (82)\\
&\equiv (((i \vee (\neg u)) \wedge (u \wedge q)) \wedge ((\neg u) \vee (\neg(\neg q)))) && \text{De Morgan} && (83)\\
&\equiv (((i \vee (\neg u)) \wedge (u \wedge q)) \wedge ((\neg u) \vee q)) && \text{Double Negation} && (84)\\
&\equiv ((i \vee (\neg u)) \wedge (u \wedge (q \wedge ((\neg u) \vee q)))) && \text{Associativity} && (85)\\
&\equiv ((i \vee (\neg u)) \wedge (u \wedge q)) && \text{Simplification 2} && (86)\\
&\equiv ((i \vee (\neg u)) \wedge u) \wedge q) && \text{Associativity} && (87)\\
&\equiv (((i \wedge u) \vee ((\neg u) \wedge u)) \wedge q) && \text{Distributivity} && (88)\\
&\equiv (((i \wedge u) \vee \texttt{F}) \wedge q) && \text{Contradiction} && (89)\\
&\equiv ((i \wedge u) \wedge q) && \text{Simplification 1} && (90)
\end{aligned}
$$

Prove that the condition leading to P_4 is true if and only if i is false and u is true.
The condition leading to P_4:

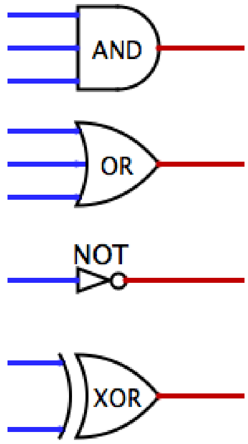
$$
\begin{aligned}
&(\neg(i \vee (\neg u))) && && (91)\\
&((\neg i) \wedge (\neg(\neg u))) && \text{De Morgan} && (92)\\
&\equiv ((\neg i) \wedge u) && \text{Double Negation} && (93)
\end{aligned}
$$

The condition leading to P_1:

$$
\begin{aligned}
&((i \vee (\neg u)) \wedge (\neg(u \wedge q))) && && (94)\\
&\equiv ((i \vee (\neg u)) \wedge ((\neg u) \vee (\neg q))) && \text{De Morgan} && (95)\\
&\equiv ((\neg u) \vee (i \wedge (\neg q))) && \text{Distributivity} && (96)
\end{aligned}
$$

1.6 Circuit Design

Basic gates:



Problem: Your instructors, Alice, Carmen, and Collin, are choosing questions to be put on the midterm. For each problem, each instructor votes either yes or not. A question is chosen if it receives two or more yes votes. Design a circuit, which outputs yes whenever a question is chosen.

1. Draw the truth table based on the problem description.

x	y	z	output
T	T	T	T
T	T	F	T
T	F	T	T
T	F	F	F
F	T	T	T
F	T	F	F
F	F	T	F
F	F	F	F

2. Convert the truth table into a propositional formula.

3. Then, convert the formula to a circuit.

Solution 1:

1. Convert the truth table into a propositional formula.

   Convert each row of the truth table to a conjunction.

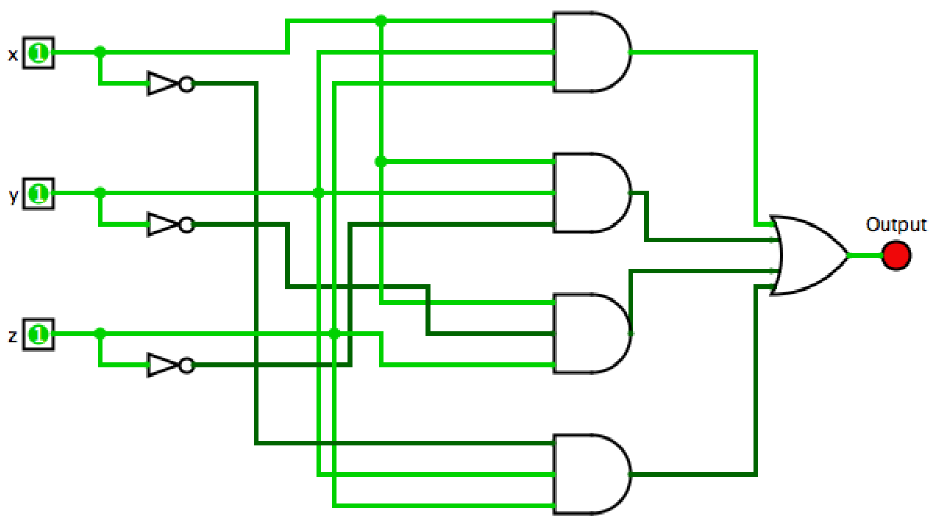
   If a variable is true in that row, write it down. Otherwise, if the variable is false, write down its negation. Then connect all variables or their negations together using AND.

   - $((x \wedge y) \wedge z)$
   - $((x \wedge y) \wedge (\neg z))$
   - $((x \wedge (\neg y)) \wedge z)$
   - $(((\neg x) \wedge y) \wedge z)$

   Connect all formulas into a disjunction.

   $(((((x \wedge y) \wedge z) \vee ((x \wedge y) \wedge (\neg z))) \vee ((x \wedge (\neg y)) \wedge z)) \vee (((\neg x) \wedge y) \wedge z))$

2. Draw the circuit.



   Making a circuit clear and readable can be challenging. Here are some advice on drawing circuits:

   - Determine where to put the inputs and the outputs first.
   - Determine where to put the major gates (the OR at the end, and one AND for each scenario).
   - Try to draw wires horizontally or vertically, not at an angle.
   - Indicate clearly whether two crossing wires are connected or not.

Solution 2:

1. Convert the truth table into a propositional formula.

   Converts rows 1-3 to a propositional formula.

   $(x \wedge (y \vee z))$

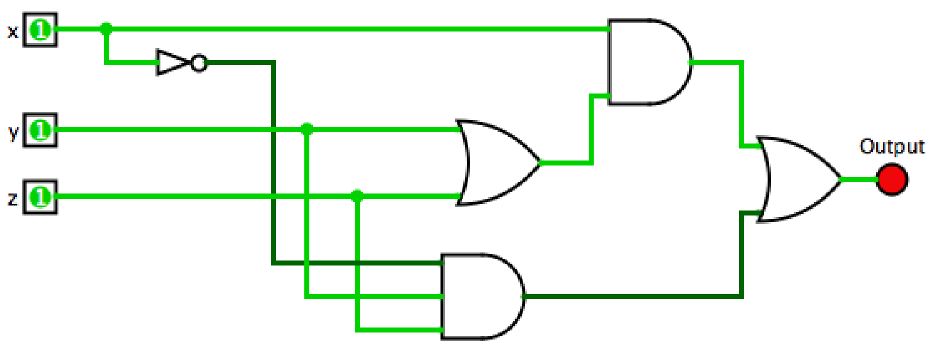
   Convert row 5 to a propositional formula.

   $(((\neg x) \wedge y) \wedge z)$

   Connect all formulas into a disjunction.

   $((x \wedge (y \vee z)) \vee (((\neg x) \wedge y) \wedge z))$

2. Draw the circuit.



Solution 3:

1. Convert the truth table into a propositional formula.

   Convert rows 1 and 5 into a propositional formula.

   $(y \wedge z)$

   Convert rows 2 and 3 into a propositional formula.

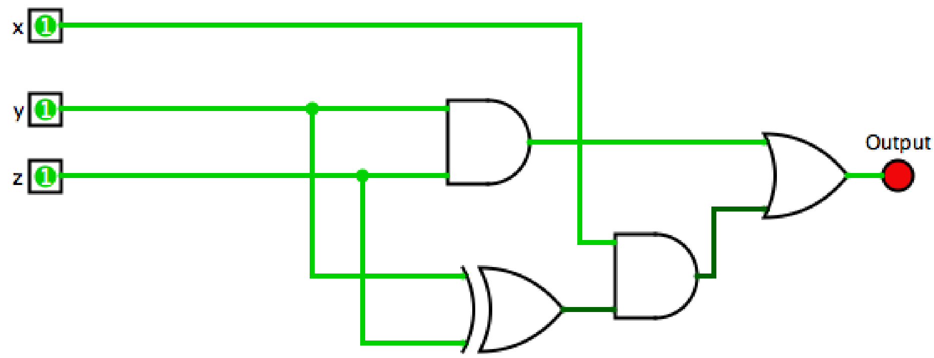
   $(x \wedge (y \oplus z))$

   For convenience, we will use the symbol $\oplus$ to represent an exclusive OR. However, you are only allowed to use this symbol in circuit design problems. You are not allowed to use this symbol for other problems because it is not a basic connective based on the definition of well-formed formulas.

   Connect all formulas into a disjunction.

   $((y \wedge z) \vee (x \wedge (y \oplus z)))$

2. Draw the circuit.



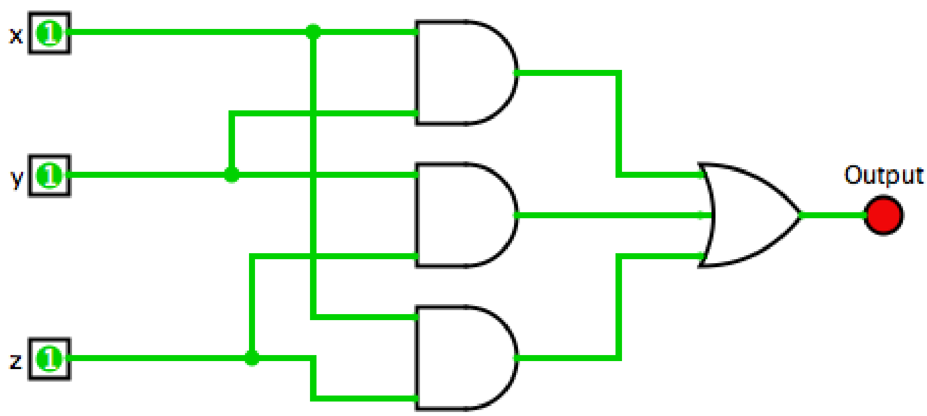
Solution 4 (contributed by Triman Kandola)

1. Convert the truth table into a propositional formula.

   $(((x \wedge y) \vee (x \wedge z)) \vee (y \wedge z))$

   This formula intuitively makes sense. If two people are already voting yes, then I don't care about what the third vote is.

2. Draw the circuit.



1.7 Semantic Entailment

Exercise 9. *Let* $\Sigma = \{(p \to q), (q \to r)\}$*. Is* Σ *satisfiable? Why or why not?*

Σ is satisfied by the truth valuation t where $t(p) = \texttt{T}$, $t(q) = \texttt{T}$ and $t(r) = \texttt{T}$.
Note that $(p \to q)^t = \texttt{T}$ and $(q \to r)^t = \texttt{T}$. Thus, Σ is satisfiable.

Exercise 10. *Let* $\Sigma = \emptyset$*. Is* Σ *satisfiable? Why or why not?*

Σ is satisfiable. In fact, any truth valuation satisfies Σ.
A truth valuation t satisfies Σ if and only if, for any formula α, if α is in Σ, then $\alpha^t = \texttt{T}$.
Since $\Sigma = \emptyset$, no formula is in Σ. The premise of the implication is false for any α, so the implication is true for every α. Therefore, any truth valuation satisfies $\Sigma = \emptyset$.

Exercise 11. *Let* $\Sigma = \{p, (\neg p)\}$*. Is* Σ *satisfiable? Why or why not?*

Σ is not satisfiable. To show this, we need to show that, under every truth valuation, at least one formula in Σ is false.

Consider an arbitrary truth valuation t. Under t, p is either true or false.

- If $p^t = \texttt{T}$, then $(\neg p)^t = \texttt{F}$. t does not satisfy Σ.
- If $p^t = \texttt{F}$, then t does not satisfy Σ.

In both cases, t does not satisfy Σ. Therefore, no truth valuation can satisfy Σ. Σ is not satisfiable.

Exercise 12. *Prove that* $\{(\neg(\mathrm{p} \wedge \mathrm{q})), (\mathrm{p} \rightarrow \mathrm{q})\} \models (\neg \mathrm{p})$.

Proof. Consider a truth valuation t such that $(\neg(\mathrm{p} \wedge \mathrm{q}))^{\mathrm{t}} = \mathtt{T}$ and $(\mathrm{p} \rightarrow \mathrm{q})^{\mathrm{t}} = \mathtt{T}$.

Since $(\mathrm{p} \rightarrow \mathrm{q})^{\mathrm{t}} = \mathtt{T}$, it is not the case that $\mathrm{p}^{\mathrm{t}} = \mathtt{T}$ and $\mathrm{q}^{\mathrm{t}} = \mathtt{F}$.
Since $(\neg(\mathrm{p} \wedge \mathrm{q}))^{\mathrm{t}} = \mathtt{T}$, it is not the case that $\mathrm{p}^{\mathrm{t}} = \mathtt{T}$ and $\mathrm{q}^{\mathrm{t}} = \mathtt{T}$.

Thus, the two premises are true under two scenarios:

- $\mathrm{p}^{\mathrm{t}} = \mathtt{F}$ and $\mathrm{q}^{\mathrm{t}} = \mathtt{T}$: In this case, $(\neg \mathrm{p})^{\mathrm{t}} = \mathtt{T}$.
- $\mathrm{p}^{\mathrm{t}} = \mathtt{F}$ and $\mathrm{q}^{\mathrm{t}} = \mathtt{F}$: In this case, $(\neg \mathrm{p})^{\mathrm{t}} = \mathtt{T}$.

In both scenarios, the conclusion is true. Thus, the entailment holds. □

Exercise 13. *Prove that* $\{(\neg(\mathrm{p} \wedge \mathrm{q})), (\mathrm{p} \rightarrow \mathrm{q})\} \not\models (\mathrm{p} \leftrightarrow \mathrm{q})$.

Proof. Consider the truth valuation t where $\mathrm{p}^{\mathrm{t}} = \mathtt{F}$ and $\mathrm{q}^{\mathrm{t}} = \mathtt{T}$.

By definitions of the connectives, $(\neg(\mathrm{p} \wedge \mathrm{q}))^{\mathrm{t}} = \mathtt{T}$, $(\mathrm{p} \rightarrow \mathrm{q})^{\mathrm{t}} = \mathtt{T}$ and $(\mathrm{p} \leftrightarrow \mathrm{q})^{\mathrm{t}} = \mathtt{F}$. Thus, the entailment does not hold. □

Exercise 14. *Prove that* $\emptyset \models ((\mathrm{p} \wedge \mathrm{q}) \rightarrow \mathrm{p}))$.

Proof. Since there is no premise, we need to prove that the conclusion $((\mathrm{p} \wedge \mathrm{q}) \rightarrow \mathrm{p}))$ is a tautology.
Consider any truth valuation t. Under t, p must be either true or false.

- $\mathrm{p}^{\mathrm{t}} = \mathtt{T}$: The conclusion of the implication $((\mathrm{p} \wedge \mathrm{q}) \rightarrow \mathrm{p}))$is true. Therefore, the implication is true.
- $\mathrm{p}^{\mathrm{t}} = \mathtt{F}$: The premise of the implication $((\mathrm{p} \wedge \mathrm{q}) \rightarrow \mathrm{p}))$ is true. Therefore, the implication is true.

Thus, the conclusion is true under any truth valuation and is a tautology. The entailment holds. □

Exercise 15. *Prove that* $\{r, (p \to (r \to q))\} \models (p \to (q \land r))$.

Proof. Consider a truth valuation t where $r^t = \mathtt{T}$ and $(p \to (r \to q))^t = \mathtt{T}$. We need to show that $(p \to (q \land r))^t = \mathtt{T}$.

Consider two cases: $p^t = \mathtt{F}$ and $p^t = \mathtt{T}$.

If $p^t = \mathtt{F}$, then $(p \to (q \land r))^t = \mathtt{T}$.

Otherwise, suppose that $p^t = \mathtt{T}$. We need to show that $(q \land r)^t = \mathtt{T}$.
By the definition of implication, $(r \to q)^t = \mathtt{T}$ since $(p \to (r \to q))^t = \mathtt{T}$. Since $r^t = \mathtt{T}$ and $(r \to q)^t = \mathtt{T}$, then $q^t = \mathtt{T}$ by the definition of implication. By the definition of $\land$, $(q \land r)^t = \mathtt{T}$ since q and r are both true under t. Therefore, $(p \to (q \land r))^t = \mathtt{T}$.

In both cases, the conclusion is true under t. The entailment holds. □

Exercise 16. *Prove that* $\{(\neg p), (q \to p)\} \not\models ((\neg p) \land q)$.

Note 1. *We need to come up with a truth valuation under which both premises are true and the conclusion is false.*
$(\neg p)$ *has to be true. So* p *has to be false under this truth valuation.*
$(q \to p)$ *has to be true and* p *is false. Thus,* q *must be false under this truth valuation.*
Therefore, this truth valuation must make p *false and* q *false.*

Proof. Consider the truth valuation t where $p^t = \mathtt{F}$ and $q^t - \mathtt{F}$.

Under this truth valuation, $(\neg p)^t = \mathtt{T}$ and $(q \to p)^t = \mathtt{T}$. Both premises are true.

Under this truth valuation, $((\neg p) \land q)^t = \mathtt{F}$. The conclusion is false.

Therefore, the entailment does not hold. □

Exercise 17. *Prove that* $\{p, (\neg p)\} \models r$.

Proof. Consider any truth valuation t under which both premises are true. If such a truth valuation exists, we have to show that r must be true under this truth valuation.

However, such a truth valuation does not exist. There are two possible cases. p is true or p is false. If p is false, then this truth valuation does not satisfy the first premise. If p is true under this truth valuation, then $(\neg p)$ must be false. This truth valuation does not satisfy the second premise.

Since no truth valuation satisfies both premises, the entailment holds. □

1.8 Natural Deduction

1.8.1 Strategies for writing a natural deduction proof

General strategies:

- Write down all of the premises
- Leave plenty of space. Then write down the conclusion.
- Look at the conclusion carefully. What is the structure of the conclusion (what is the last connective applied in the formula? Can you apply an introduction rule to produce the conclusion?
- Look at each premise carefully. What is the structure of the premise (what is the last connective applied in the formula)? Can you apply an elimination rule to simplify it?
- Working backwards from the conclusion is often more effective than working forward from the premises. It keeps your eyes on the prize.
- If no rule is applicable, consider using a combination of ¬i and ¬¬e.

Working with subproofs

- To apply an introduction rule to produce the conclusion, **lay down the structure of the subproof before you proceed to fill in the subproof**. That is, draw the box for the subproof, write down the assumption on the first line, copy the conclusion to the last line of the subproof.
- Every subproof must be created to apply a particular rule. If you don't know what rule you are trying to apply, don't create a subproof.
- When filling in a subproof, you can use all the formulas that come before as long as the formula is not in a previous subproof that has already closed.
- Outside of a subproof, you have to use the subproof as a whole. You cannot use any individual formula in the subproof.

1.8.2 And elimination and introduction

Exercise 18. *Show that* $\{(\mathrm{p} \wedge \mathrm{q}), (\mathrm{r} \wedge \mathrm{s})\} \vdash (\mathrm{q} \wedge \mathrm{s})$.

1.	$(\mathrm{p} \wedge \mathrm{q})$	premise
2.	$(\mathrm{r} \wedge \mathrm{s})$	premise
3.	q	$\wedge$e: 1
4.	s	$\wedge$e: 2
5.	$(\mathrm{q} \wedge \mathrm{s})$	$\wedge$i: 3, 4

Exercise 19. *Show that* $((\mathrm{p} \wedge \mathrm{q}) \wedge \mathrm{r}) \vdash (\mathrm{p} \wedge (\mathrm{q} \wedge \mathrm{r}))$.

1.	$((\mathrm{p} \wedge \mathrm{q}) \wedge \mathrm{r})$	premise
2.	$(\mathrm{p} \wedge \mathrm{q})$	$\wedge$e: 1
3.	p	$\wedge$e: 2
4.	q	$\wedge$e: 2
5.	r	$\wedge$e: 1
6.	$(\mathrm{q} \wedge \mathrm{r})$	$\wedge$i; 4, 5
7.	$(\mathrm{p} \wedge (\mathrm{q} \wedge \mathrm{r}))$	$\wedge$i: 3, 6

1.8.3 Implication introduction and elimination

Exercise 20. *Show that* $\{(\mathrm{p} \to \mathrm{q}), (\mathrm{q} \to \mathrm{r})\} \vdash (\mathrm{p} \to \mathrm{r})$.

1.	$(\mathrm{p} \to \mathrm{q})$	premise
2.	$(\mathrm{q} \to \mathrm{r})$	premise
3.	p	assumption
4.	q	$\to$e: 1, 3
5.	r	$\to$e: 2, 4
6.	$(\mathrm{p} \to \mathrm{r})$	$\to$i: 3–5

Exercise 21. *Show that* $\{(\mathrm{p} \to (\mathrm{q} \to \mathrm{r})), (\mathrm{p} \to \mathrm{q})\} \vdash (\mathrm{p} \to \mathrm{r})$.

1.	$(\mathrm{p} \to (\mathrm{q} \to \mathrm{r}))$	premise
2.	$(\mathrm{p} \to \mathrm{q})$	premise
3.	p	assumption
4.	$(\mathrm{q} \to \mathrm{r})$	$\to$e: 1, 3
5.	q	$\to$e: 2, 3
6.	r	$\to$e: 4, 5
7.	$(\mathrm{p} \to \mathrm{r})$	$\to$i: 3-6

Exercise 22. *Show that* $\{(p \to (q \to r))\} \vdash ((p \land q) \to r)$.

1.	$(p \to (q \to r))$	premise
2.	$(p \land q)$	assumption
3.	p	$\land$e: 2
4.	$(q \to r)$	$\to$e: 1, 3
5.	q	$\land$e: 2
6.	r	$\to$e: 4, 5
7.	$((p \land q) \to r)$	$\to$i: 2–6

Exercise 23. *Show that* $\{((p \land q) \to r)\} \vdash (p \to (q \to r))$.

1.	$((p \land q) \to r)$	premise
2.	p	assumption
3.	q	assumption
4.	$(p \land q)$	$\land$i: 2, 3
5.	r	$\to$e: 1, 4
6.	$(q \to r)$	$\to$i: 3–5
7.	$(p \to (q \to r))$	$\to$i: 2–6

1.8.4 Or elimination and introduction

Exercise 24. *Show that* $\{(p \wedge (q \vee r))\} \vdash ((p \wedge q) \vee (p \wedge r))$.

1.	$(p \wedge (q \vee r))$	premise
2.	p	$\wedge$e: 1
3.	$(q \vee r)$	$\wedge$e: 1
4.	q	assumption
5.	$(p \wedge q)$	$\wedge$i: 2, 4
6.	$((p \wedge q) \vee (p \wedge r))$	$\vee$i: 5
7.	r	assumption
8.	$(p \wedge r)$	$\wedge$i: 2, 7
9.	$((p \wedge q) \vee (p \wedge r))$	$\vee$i: 8
10.	$((p \wedge q) \vee (p \wedge r))$	$\vee$e: 3, 4–6, 7–9

Exercise 25. *Show that* $\{((p \wedge q) \vee (p \wedge r))\} \vdash (p \wedge (q \vee r))$.

1.	$((p \wedge q) \vee (p \wedge r))$	premise
2.	$(p \wedge q)$	assumption
3.	p	$\wedge$e: 2
4.	q	$\wedge$e: 2
5.	$(q \vee r)$	$\vee$i: 4
6.	$(p \wedge (q \vee r))$	$\wedge$i: 3, 5
7.	$(p \wedge r)$	assumption
8.	p	$\wedge$e: 7
9.	r	$\wedge$e: 7
10.	$(q \vee r)$	$\vee$i: 9
11.	$(p \wedge (q \vee r))$	$\wedge$i: 8, 10
12.	$(p \wedge (q \vee r))$	$\vee$e: 1, 2-6, 7-11

Exercise 26. *Show that* $\{(p \lor q)\} \vdash ((p \to q) \lor (q \to p))$.

1.	$(p \lor q)$	premise
2.	p	assumption
3.	q	assumption
4.	p	Reflexivity: 2
5.	$(q \to p)$	→i: 3–4
6.	$((p \to q) \lor (q \to p))$	∨i: 5
7.	q	assumption
8.	p	assumption
9.	q	Reflexivity: 7
10.	$(p \to q)$	→i: 8–9
11.	$((p \to q) \lor (q \to p))$	∨i: 10
12.	$((p \to q) \lor (q \to p))$	∨e: 1, 2–6, 7–11

Exercise 27. *Show that* $\{(p \to q)\} \vdash ((r \lor p) \to (r \lor q))$.

1.	$(p \to q)$	premise
2.	$(r \lor p)$	assumption
3.	r	assumption
4.	$(r \lor q)$	∨i: 3
5.	p	assumption
6.	q	→e: 1, 5
7.	$(r \lor q)$	∨i: 6
8.	$(r \lor q)$	∨e: 2, 3–4, 5–7
9.	$((r \lor p) \to (r \lor q))$	→e: 2–8

1.8.5 Negation introduction and double negation elimination

Exercise 28. *Show that* $\{(p \to (\neg p))\} \vdash (\neg p)$.

1.	$(p \to (\neg p))$	premise
2.	p	assumption
3.	$(\neg p)$	$\to$e: 1, 2
4.	$\bot$	$\bot$i: 2, 3
5.	$(\neg p)$	$\neg$i, 2–4

Exercise 29. *Show that* $\{(p \to (q \to r)), p, (\neg r)\} \vdash (\neg q)$.

1.	$(p \to (q \to r))$	premise
2.	p	premise
3.	$(\neg r)$	premise
4.	$(q \to r)$	$\to$e: 1, 2
5.	q	assumption
6.	r	$\to$e: 4, 5
7.	$\bot$	$\bot$i: 3, 6
8.	$(\neg q)$	$\neg$i: 5–7

Exercise 30. *Show that* $\{((\neg p) \to (\neg q))\} \vdash (q \to p)$.

1.	$((\neg p) \to (\neg q))$	premise
2.	q	assumption
3.	$(\neg p)$	assumption
4.	$(\neg q)$	$\to$e: 1, 3
5.	$\bot$	$\bot$i: 2, 4
6.	$(\neg(\neg p))$	$\neg$i: 3-5
7.	p	$\neg\neg$e: 6
8.	$(q \to p)$	$\to$i: 2-7

Exercise 31. *Show that* $\{((p \wedge (\neg q)) \to r), (\neg r), p\} \vdash q$.

1.	$((p \wedge (\neg q)) \to r)$	premise
2.	$(\neg r)$	premise
3.	p	premise
4.	$(\neg q)$	assumption
5.	$(p \wedge (\neg q))$	$\wedge$i: 3, 4
6.	r	$\to$e: 1, 5
7.	$\bot$	$\bot$i: 2, 6
8.	$(\neg(\neg q))$	$\neg$i: 4–7
9.	q	$\neg\neg$e: 8

1.8.6 Negation elimination

Exercise 32. *Show that* $\{(\mathrm{p} \vee \mathrm{q}), (\neg \mathrm{p})\} \vdash \mathrm{q}$.

1.	$(\mathrm{p} \vee \mathrm{q})$	premise
2.	$(\neg \mathrm{p})$	premise
3.	p	assumption
4.	$\bot$	$\bot$i: 2, 3
5.	q	$\bot$e: 4
6.	q	assumption
7.	q	$\vee$e: 1, 3-5, 6

Exercise 33. *Show that* $\emptyset \vdash ((\neg \mathrm{p}) \rightarrow (\mathrm{p} \rightarrow (\mathrm{p} \rightarrow \mathrm{q})))$.

1.	$(\neg \mathrm{p})$	assumption
2.	p	assumption
3.	$\bot$	$\bot$i: 1, 2
4.	$(\mathrm{p} \rightarrow \mathrm{q})$	$\bot$e: 3
5.	$(\mathrm{p} \rightarrow (\mathrm{p} \rightarrow \mathrm{q}))$	$\rightarrow$i: 2-4
6.	$((\neg \mathrm{p}) \rightarrow (\mathrm{p} \rightarrow (\mathrm{p} \rightarrow \mathrm{q})))$	$\rightarrow$i: 1-5

1.8.7 Putting it together!

Exercise 34. *(De Morgan's Law) Show that* $\{(\neg(\alpha \vee \beta))\} \vdash ((\neg\alpha) \wedge (\neg\beta))$.

1.	$(\neg(\alpha \vee \beta))$	premise
2.	α	assumption
3.	$(\alpha \vee \beta)$	$\vee$i: 2
4.	$\bot$	$\bot$i: 1, 3
5.	$(\neg\alpha)$	$\neg$i: 2–4
6.	β	assumption
7.	$(\alpha \vee \beta)$	$\vee$i: 6
8.	$\bot$	$\bot$i: 1, 7
9.	$(\neg\beta)$	$\neg$i: 6–8
10.	$((\neg\alpha) \wedge (\neg\beta))$	$\wedge$i: 5, 9

Exercise 35. *(De Morgan's Law) Show that* $\{((\neg\alpha) \wedge (\neg\beta))\} \vdash (\neg(\alpha \vee \beta))$.

1.	$((\neg\alpha) \wedge (\neg\beta))$	premise
2.	$(\alpha \vee \beta)$	assumption
3.	α	assumption
4.	$(\neg\alpha)$	$\wedge$e: 1
5.	$\bot$	$\bot$i: 3, 4
6.	β	assumption
7.	$(\neg\beta)$	$\wedge$e: 1
8.	$\bot$	$\bot$i: 7, 8
9.	$\bot$	$\vee$e: 2, 3–6, 7–10
10.	$(\neg(\alpha \vee \beta))$	$\neg$i: 2–11

Exercise 36. *(De Morgan's Law) Show that* $\{((\neg\alpha) \vee (\neg\beta))\} \vdash (\neg(\alpha \wedge \beta))$.

1.	$((\neg\alpha) \vee (\neg\beta))$	premise
2.	$(\neg\alpha)$	assumption
3.	$(\alpha \wedge \beta)$	assumption
4.	α	$\wedge$e: 3
5.	$\bot$	$\bot$i: 2, 4
6.	$(\neg(\alpha \wedge \beta))$	$\neg$i: 3-5
7.	$(\neg\beta)$	assumption
8.	$(\alpha \wedge \beta)$	assumption
9.	β	$\wedge$e: 8
10.	$\bot$	$\bot$i: 7, 9
11.	$(\neg(\alpha \wedge \beta))$	$\neg$i: 8-10
12.	$(\neg(\alpha \wedge \beta))$	$\vee$e: 1, 2-6, 7-11

Exercise 37. *(De Morgan's Law) Show that* $\{(\alpha \vee \beta)\} \vdash (\neg((\neg\alpha) \wedge (\neg\beta)))$.

1.	$(\alpha \vee \beta)$	premise
2.	α	assumption
3.	$((\neg\alpha) \wedge (\neg\beta))$	assumption
4.	$(\neg\alpha)$	$\wedge$e: 3
5.	$\bot$	$\bot$i: 2, 4
6.	$(\neg((\neg\alpha) \wedge (\neg\beta)))$	$\neg$i: 3-5
7.	β	assumption
8.	$((\neg\alpha) \wedge (\neg\beta))$	assumption
9.	$(\neg\beta)$	$\wedge$e: 8
10.	$\bot$	$\bot$i: 7, 9
11.	$(\neg((\neg\alpha) \wedge (\neg\beta)))$	$\neg$i: 8-10
12.	$(\neg((\neg\alpha) \wedge (\neg\beta)))$	$\vee$e: 1, 2-6, 7-11

Exercise 38. *(De Morgan's Law) Show that* $\{(\neg(\alpha \wedge \beta))\} \vdash ((\neg\alpha) \vee (\neg\beta))$.

1.	$(\neg(\alpha \wedge \beta))$	premise
2.	$(\neg((\neg\alpha) \vee (\neg\beta)))$	assumption
3.	$(\neg\alpha)$	assumption
4.	$((\neg\alpha) \vee (\neg\beta))$	$\vee$i: 3
5.	$\bot$	$\bot$i: 2, 4
6.	α	PBC: 3-5
7.	$(\neg\beta)$	assumption
8.	$((\neg\alpha) \vee (\neg\beta))$	$\vee$i: 7
9.	$\bot$	$\bot$i: 2, 8
10.	β	PBC: 7-9
11.	$(\alpha \wedge \beta)$	$\wedge$i: 6, 10
12.	$\bot$	$\bot$i: 1, 11
13.	$((\neg\alpha) \vee (\neg\beta))$	PBC: 2-12

Exercise 39. *Show that* $\{(\neg(p \to q))\} \vdash (q \to p)$.

1.	$(\neg(p \to q))$	premise
2.	q	assumption
3.	$(\neg p)$	assumption
4.	p	assumption
5.	q	reflexive: 2
6.	$(p \to q)$	$\to$i: 4-5
7.	$\bot$	$\bot$i: 1, 6
8.	$(\neg(\neg p))$	$\neg$i: 3-7
9.	p	$\neg\neg$e: 8
10.	$(q \to p)$	$\to$i: 2-9

1.8.8 Other problems

Exercise 40. *E4 Exercise 4: Prove that for any set of propositional formulas* Σ *and any propositional variables* p *and* q*, if* $\Sigma \vdash$ p*, then* $\Sigma \vdash ((\neg p) \to q)$*.*

Proof. Let Σ be a set of propositional formulas and let p and q be propositional variables. Assume that $\Sigma \vdash$ p. This means that the following proof exists.

1.	Σ	premises
2.	...	...
3.	p	

Using the above proof, we will construct a natural deduction proof for $\Sigma \vdash ((\neg p) \to q)$.

1.	Σ	premises
2.	...	...
3.	p	
4.	$(\neg p)$	assumption
5.	$\perp$	$\perp$i: 3, 4
6.	q	$\perp$e: 5
7.	$((\neg p) \to q)$	$\to$i: 4-6

Therefore, $\Sigma \vdash ((\neg p) \to q)$ holds. □

1.9 Soundness and Completeness of Natural Deduction

1.9.1 The soundness of inference rules

Exercise 41. *The following inference rule is called Disjunctive syllogism.*

$$\frac{(\neg\alpha) \quad (\alpha \vee \beta)}{\beta} \textit{ Disjunctive syllogism}$$

where α *and* β *are well-formed propositional formulas.*

Prove that this inference rule is sound. That is, prove that the following semantic entailment holds.

$$\{(\neg\alpha), (\alpha \vee \beta)\} \vDash \beta$$

You must use **the definition of semantic entailment** *to write your proof. Do not use any other technique such as truth table, valuation tree, logical identities, natural deduction, soundness, or completeness.*

Note 2. *To prove that an entailment holds, we need to consider all truth valuations under which all of the premises are true. For each such truth valuation, we need to show that the conclusion is true.*

The proof typically looks like the following.

- *Consider a truth valuation* t *under which all of the premises are true.*
- *If premise 1 is true under* t*, then* α *must be ... under* t *and* β *must be ... under* t*. If premise 2 is true under* t*, then ...*
- *There are ... cases that we need to consider.*
- *Case 1: this case is impossible because .../... the conclusion is true under* t*.*
- *Case 2: ...*
- *The conclusion is true in every case. Therefore, the entailment holds.*

Proof. Consider a truth valuation t under which $(\neg\alpha)^t = \mathtt{T}$ and $(\alpha \vee \beta)^t = \mathtt{T}$. We need to show that $\beta^t = \mathtt{T}$.
Since $(\alpha \vee \beta)^t = \mathtt{T}$, at least one of α and β is true under t.
Since $(\neg\alpha)^t = \mathtt{T}$, α is false under t. Therefore, β must be true under t.
Therefore, the entailment holds. □

Exercise 42. *Consider the following inference rule:*

$$\frac{(\alpha \to \beta)}{(\beta \to \alpha)} \textit{ Flip the implication}$$

where α *and* β *are well-formed propositional formulas.*

Prove that this inference rule is NOT sound. That is, prove the following statement:

$$\{(\alpha \to \beta)\} \not\models (\beta \to \alpha)$$

You must use **the definition of semantic entailment** *to write your proof. Do not use any other technique such as truth table, valuation tree, logical identities, natural deduction, soundness, or completeness.*

Note 3. *To prove that an entailment does not hold, we need to find a concrete counterexample, which shows that, there is a truth valuation* t *under which all of the premises are true and the conclusion is false.*

A concrete counterexample consist of the following:

- *Choose concrete formulas for* α *and* β*. In the following proof, we let* α *be* p *and* β *be* q *where* p *and* q *are propositional variables.*
- *Choose a truth valuation* t *such that all the premises are true and the conclusion is false.*

Choosing a concrete formula for each symbol is important. In the proof below, if we do not assign concrete formulas to α *and* β*, then we cannot make claims about their truth values under* t*. We want to find a truth valuation under which* β *is true and* α *is false. This is not possible if* β *is* $(r \wedge (\neg r))$ *and* α *is* $(r \vee (\neg r))$*.*

The difficult part is coming up with a counterexample that works. After that, writing up the proof is straightforward.

Proof. To prove that the entailment does not hold, we need to find one counterexample.

Let p and q be two propositional variables. Let α be p and let β be q. Consider a truth valuation t under which $p^t = \mathtt{F}$ and $q^t = \mathtt{T}$.

Under t, the premise is true. $(\alpha \to \beta)^t = (p \to q)^t = \mathtt{T}$.
Under t, the conclusion is false. $(\beta \to \alpha)^t = (q \to p)^t = \mathtt{F}$.

We found a truth valuation under which the premise is true and the conclusion is false. Thus, the entailment does not hold. □

1.9.2 Soundness and Completeness of Natural Deduction

Exercise 43. *Prove or disprove this statement: If* $\{a, b\} \vdash c$*, then* $\emptyset \models ((a \land b) \to c)$*.* a*,* b*, and* c *are well-formed propositional formulas.*

Note 4. *The statement is an implication, and the premise and the conclusion of the implication differ in two ways. The premise is about the existence of a natural deduction proof, whereas the conclusion is about an entailment. Moreover, the premise has* a *and* b *on the left hand side, whereas the conclusion has everything on the right hand side. Thus, there are two ways for us to transform the premise into the conclusion.*

Approach 1:

A visual representation of approach 1:

$$\{a, b\} \vdash c \quad \to \quad \{a, b\} \models c \quad \to \quad \emptyset \models ((a \land b) \to c)$$

First, we transform $\{a, b\} \vdash c$ *(the existence of a proof) to* $\{a, b\} \models c$ *(an entailment) by using the soundness of natural deduction.*

Then, we move a *and* b *from the left hand side to the right hand side by proving that* $\{a, b\} \models c$ *are* $\emptyset \models ((a \land b) \to c)$ *equivalent by the definition of entailment.*

Approach 2:

A visual representation of approach 2:

$$\{a, b\} \vdash c \quad \to \quad \emptyset \vdash ((a \land b) \to c) \quad \to \quad \emptyset \models ((a \land b) \to c)$$

First, we move a *and* b *from the left hand side to the right hand side by proving that* $\{a, b\} \vdash c$ *and* $\emptyset \vdash ((a \land b) \to c)$ *are equivalent.*

Then, we transform $\emptyset \vdash ((a \land b) \to c)$ *(the existence of a proof) to* $\emptyset \models ((a \land b) \to c)$ *(an entailment) by the soundness of natural deduction.*

See the two proofs on the following page.

Proof 1. We will prove the statement.

Assume $\{a, b\} \vdash c$ holds.

By the soundness of natural deduction, the entailment $\{a, b\} \vDash c$ holds.

Consider a truth valuation t under which $a^t = \mathtt{T}$ and $b^t = \mathtt{T}$. We know that $c^t = \mathtt{T}$ by $\{a, b\} \vDash c$. Therefore, by the definition of an implication, we know that $((a \wedge b) \to c)$ is a tautology.

Consider a truth valuation t. There is no formula in $\emptyset$. Thus, t satisfies $\emptyset$. t also satisfies $((a \wedge b) \to c)$ since $((a \wedge b) \to c)$ is a tautology. Therefore, the entailment $\emptyset \vDash ((a \wedge b) \to c)$ holds.

□

Proof 2. We will prove the statement.

Assume $\{a, b\} \vdash c$ holds. Thus, there is a natural deduction proof which starts with a and b as the premises and ends with c.

1.	a	premise
2.	b	premise
3.	...	...
4.	c	...

We construct a natural deduction proof for $\emptyset \vdash ((a \wedge b) \to c)$ as follows.

1.	$(a \wedge b)$	assumption
2.	a	$\wedge$e: 1
3.	b	$\wedge$e: 1
4.	...	...
5.	c	...
6.	$((a \wedge b) \to c)$	$\to$i: 1-5

This proof shows that $\emptyset \vdash ((a \wedge b) \to c)$ holds.

By the soundness of natural deduction, the entailment $\emptyset \vDash ((a \wedge b) \to c)$ holds.

□

Exercise 44. *Prove or disprove this statement: If* $\{\alpha\} \models \beta$*, then* $\emptyset \vdash (\beta \to \alpha)$*.* α *and* β *are well-formed propositional formulas.*

Note 5. *To show that the implication is false, we need to choose concrete formulas for* α *and* β *such that the premise is true and the conclusion is false.*

By inspecting the premise and the conclusion, we see that the concrete formulas need to make sure that α *entails* β*, but* β *does not entail* α*.*

Choosing α *to be* p *and* β *to be* $(\mathrm{p} \vee \mathrm{q})$ *satisfy both requirements.*

Proof. We will disprove the statement.

Let p and q be two propositional variables. Let α be p and let β be $(\mathrm{p} \vee \mathrm{q})$.

First, we prove that $\{\alpha\} \models \beta$ holds. Consider a truth valuation t under which α is true. This means that $\mathrm{p}^{\mathrm{t}} = \mathtt{T}$. Under t, β is true because $(\mathrm{p} \vee \mathrm{q})^{\mathrm{t}} = \mathtt{T}$. Therefore, the entailment $\{\alpha\} \models \beta$ holds.

Now, we prove that $\emptyset \nvdash (\beta \to \alpha)$ holds. To show that such a proof does not exist, it suffices to show that the corresponding entailment $\emptyset \models (\beta \to \alpha)$ does not hold. Then by the contrapositive of the soundness of natural deduction, we have that $\emptyset \nvdash (\beta \to \alpha)$ holds.

To prove that $\emptyset \not\models (\beta \to \alpha)$ (or $\emptyset \not\models ((\mathrm{p} \vee \mathrm{q}) \to \mathrm{p})$), we consider a truth valuation t such that $\mathrm{p}^{\mathrm{t}} = \mathtt{F}$ and $\mathrm{q}^{\mathrm{t}} = \mathtt{T}$. Under t, $\beta^{\mathrm{t}} = (\mathrm{p} \vee \mathrm{q})^{\mathrm{t}} = \mathtt{T}$ and $\alpha^{\mathrm{t}} = \mathrm{p}^{\mathrm{t}} = \mathtt{F}$. Therefore, $\emptyset \not\models (\beta \to \alpha)$ holds.

□

2 Predicate Logic

2.1 Translations

Exercise 45. *Let the domain be the set of animals. Let* B(x) *mean that* x *is a bear. Let* H(x) *mean that* x *likes honey.*

Translate "every bear likes honey" into predicate logic.

People often come up with the following two translations. See the formulas and the corresponding explanations below.

- $(\forall x\ (B(x) \land H(x)))$

  This formula says that every animal x is a bear and likes honey.

  This formula is an incorrect translation. The original sentence does not require every animal to be a bear. The sentence simply ignores any animal that is not a bear and focuses on animals that are bears.

- $(\forall x\ (B(x) \rightarrow H(x)))$

  This formula says that for every animal x, if x is a bear, then x likes honey.

  This is a correct translation. If an animal is a bear, then it must like honey as required by the original sentence. If an animal is not a bear, then the premise of the implication is false, which means that the implication is vacuously true. (In other words, we don't care about animals that are not bears.)

To differentiate between two predicate formulas, it is often a useful exercise to come up with a domain for which one formula is true and the other formula is false.

Consider a domain, which contains a bear A who likes honey and a rabbit B.

- For this domain, the first formula is false. When x is rabbit B, x is not a bear.
- For this domain, the second formula is true. When x is bear A, it likes honey, so the implication is true. When x is rabbit B, it is not a bear, so the implication is vacuously true. Since the implication is true for every element of the domain, the formula is true.

In general, consider a domain D and a predicate P(x).
The following sentence

> "All <things in D for which P is true> have the property Q."

translates into the formula

$$(\forall x\ (P(x) \rightarrow Q(x))).$$

Exercise 46. *Let the domain be the set of animals. Let* B(x) *mean that* x *is a bear. Let* H(x) *mean that* x *likes honey.*

Translate "some bear likes honey" into predicate logic.

People often come up with the following two translations. See the formulas and the corresponding explanations below.

- $(\exists x\ (B(x) \wedge H(x)))$

  This formula says that there is an animal x, which is a bear and likes honey.

  This formula is the correct translation. The original sentence requires that there is a bear in the domain. Furthermore, it requires that there is a bear in the domain that likes honey. This formula guarantees both.

- $(\exists x\ (B(x) \rightarrow H(x)))$

  This formula says that there is an animal x, which is either not a bear, or is a bear and likes honey.

  This sentence is an incorrect translation, although many people think that it makes intuitive sense. The problem with this formula comes from the fact that the implication is vacuously true when the premise is false. This formula does not guarantee that there has to be a bear in the domain. As soon as we find an animal that is not a bear in the domain, the premise of the implication is false and the implication is vacuously true. This does not correspond to the original sentence, which requires that there is a bear in the domain.

To differentiate these two formulas, let's consider a domain, which contains a rabbit B. For this domain, the original sentence should be false because there is no bear.

- For this domain, the first formula is false. We cannot find a bear in the domain, which is required by the formula.

- For this domain, the second formula is true. When x is rabbit B, B is not a bear, so the premise of the implication is false. Thus, the implication is vacuously true. Since we have found an animal which makes the implication true, the formula is true.

In general, consider a domain D and a predicate P(x).
The following sentence

> "Some <thing in D for which P is true> have the property Q."

translates into the formula

$$(\exists x\ (P(x) \wedge Q(x))).$$

Exercise 47. *Could you summarize the general patterns of translations based on the two exercises above? Which binary connectives usually go with the universal and the existential quantifiers?*

Alice: I put this exercise here so that I will have a place to put down a summary.

As a general rule of thumb, the universal quantifier is often used in conjunction with the implication ($\rightarrow$), and the existential quantifier is often used in conjunction with the conjunction ($\wedge$). We've seen examples of both above.

The universal quantifier

- $\forall$ and $\rightarrow$: This universal quantifier pairs well with the implication. This combination is used to make a statement about a subset of the domain. Therefore, we use the premise of the implication to restrict our attention to this subset. We don't have to worry about any element that is not in this subset because the implication is vacuously true for any such element.

- $\forall$ and $\wedge$: This combination is not impossible. However, it is a very strong statement. This combination is claiming that every element of the domain must satisfy the properties connected by the $\wedge$. If this is what you meant to express, then go ahead and use this combination.

The existential quantifier

- $\exists$ and $\wedge$: The existential quantifier pairs well with the conjunction. This combination can be used to express the fact that there exists an element of domain which has the two properties connected by the conjunction.

- $\exists$ and $\rightarrow$: This combination does not make sense logically. The main reason is that it is too easy to make such a formula true. As soon as we find an element of the domain, which makes the premise of the implication false, the implication is vacuously true and the formula is true as well.

Exercise 48. *Translate the following sentences into predicate formulas.*

Let the domain contain the set of all students and courses. Define the following predicates:
C(x): x is a course.
S(x): x is a student.
T(x, y): student x has taken course y.

1. Every student has taken some course.
   $(\forall x\ (S(x) \rightarrow (\exists y\ (C(y) \wedge T(x, y)))))$

2. A student has taken a course.
   $(\exists x\ (S(x) \wedge (\exists y\ (C(y) \wedge T(x, y)))))$

3. No student has taken every course.
   $(\neg(\exists x\ (S(x) \wedge (\forall y\ (C(y) \rightarrow T(x, y))))))$

4. Some student has not taken any course.
   $(\exists x\ (S(x) \wedge (\forall y\ (C(y) \rightarrow (\neg T(x, y))))))$

5. Every student has taken every course.
   $(\forall x\ (S(x) \rightarrow (\forall y\ (C(y) \rightarrow T(x, y)))))$

Exercise 49. *Translating "at least", "at most", and "exactly". Translate the following sentences into predicate formulas.*

- There is at least one bear.

$$(\exists x\ B(x))$$

- There are at least two bears.

$$(\exists x\ (\exists y\ ((B(x) \land B(y)) \land (x \neq y))))$$

The formula says: there are two bears x and y, and x and y must be different. Note that, if we don't have $(x \neq y)$, the formula only guarantees that there exists one bear because x and y could refer to the same animal in the domain.

- There is at most one bear.

$$(\neg(\exists x\ (\exists y\ ((B(x) \land B(y)) \land (x \neq y)))))$$

The negation of "at most one" is "at least two". Therefore, the sentence is equivalent to "It is not the case that there exist two different bears".

Using the generalized De Morgan's laws, we can show that the above formula is logically equivalent to the formula below.

$$(\forall x\ (\forall y\ ((B(x) \land B(y)) \rightarrow (x = y))))$$

This formula says that: If we can find two bears x and y, then x and y must refer to the same bear. To understand this formula, imagine that I made the claim that there is at most one bear. Then your goal is to disprove my claim. You find two bears in the domain and show them to me. For my claim to be true, I have to be able to prove that the two bears you found are actually the same bear. I have to be able to do this no matter which two bears you show to me.

Yet another translation is that: ((there is no bear) or (there is exactly one bear)). We can use any translation of "there is exactly one bear" on the next page.

$$((\forall x\ (\neg B(x))) \lor (\exists y\ (B(y) \land (\forall z\ (B(z) \rightarrow (y = z))))))$$

- There is exactly one bear.

  One translation is: there is at least one bear and there is at most one bears.

  $$((\exists z\ B(z)) \wedge ((\neg(\exists x\ (\exists y\ ((B(x) \wedge B(y)) \wedge (x \neq y)))))))$$

  Another translation: there is at least one bear and if there is another bear, then the two bears must be the same.

  $$(\exists x\ (B(x) \wedge (\forall y\ (B(y) \rightarrow (x = y)))))$$

2.2 Semantics of Predicate Formulas

Consider this language of predicate logic:

- Constant symbols: $\mathrm{a}, \mathrm{b}, \mathrm{c}$
- Variable symbols: $\mathrm{x}, \mathrm{y}, \mathrm{z}$
- Function symbols: $\mathrm{f}^{(1)}, \mathrm{g}^{(2)}$
- Predicate symbols: $\mathrm{P}^{(1)}, \mathrm{Q}^{(2)}$

2.2.1 Evaluating Formulas with No Variables

Exercise 50. *Give an interpretation* I *such that* $\mathrm{Q}(\mathrm{f}(\mathrm{c}), \mathrm{a})^{\mathrm{I}} = \mathtt{T}$ *where* $\mathrm{dom}(\mathrm{I}) = \{1, 2, 3\}$.

Note 6. *I don't like to work with weird functions. So let's fix the function* f *to something simple first. Let* f^{I} *be* $\mathrm{f}^{\mathrm{I}}(\mathrm{x}) = \mathrm{x}, \forall \mathrm{x} \in \mathrm{dom}(\mathrm{I})$. *Given this, we simplify the formula below.*

$$\mathrm{f}(\mathrm{c})^{\mathrm{I}} = \mathrm{f}^{\mathrm{I}}(\mathrm{c}^{\mathrm{I}}) = \mathrm{c}^{\mathrm{I}} \tag{97}$$

$$\mathrm{Q}(\mathrm{f}(\mathrm{c}), \mathrm{a})^{\mathrm{I}} = \mathrm{Q}(\mathrm{c}, \mathrm{a})^{\mathrm{I}} \tag{98}$$

The function g *does not appear in the formula. Nevertheless, we need to give it an interpretation. Let* g^{I} *be* $\mathrm{g}^{\mathrm{I}}(\mathrm{x}) = 1, \forall \mathrm{x} \in \mathrm{dom}(\mathrm{I})$.

I like to deal with the predicates last. So let's assign meanings to the constant symbols. Let $\mathrm{c}^{\mathrm{I}} = 1$ *and* $\mathrm{a}^{\mathrm{I}} = 2$. *Then, we have that* $\mathrm{Q}(\mathrm{c}, \mathrm{a})^{\mathrm{I}}$ *is true if and only if* $\langle 1, 2\rangle \in \mathrm{Q}^{\mathrm{I}}$.
The constant symbol b *does not appear in the formula, but we still need to give it an interpretation. Let* $\mathrm{b}^{\mathrm{I}} = 1$.

Finally, let's define Q^{I}. *Above the above analysis, at a minimum, we need* $\langle 1, 2\rangle \in \mathrm{Q}^{\mathrm{I}}$. *We could include other tuples in* Q^{I} *if we like, but they don't affect the truth value of this formula. Thus, let* $\mathrm{Q}^{\mathrm{I}} = \{\langle 1, 2\rangle\}$.
P *does not appear in the formula, but we still need to give it an interpretation. Let* P^{I} *be the empty set.*

Solution: The interpretation I is given below.

- $\mathrm{dom}(\mathrm{I}) = \{1, 2, 3\}$.
- $\mathrm{a}^{\mathrm{I}} = 2, \mathrm{b}^{\mathrm{I}} = 1, \mathrm{c}^{\mathrm{I}} = 1$.
- $\mathrm{f}^{\mathrm{I}}(\mathrm{x}) = \mathrm{x}, \forall \mathrm{x} \in \mathrm{dom}(\mathrm{I})$, $\mathrm{g}^{\mathrm{I}}(\mathrm{x}) = 1, \forall \mathrm{x} \in \mathrm{dom}(\mathrm{I})$.
- $\mathrm{Q}^{\mathrm{I}} = \{\langle 1, 2\rangle\}$, $\mathrm{P}^{\mathrm{I}} = \emptyset$.

Therefore, $\mathrm{Q}(\mathrm{f}(\mathrm{c}), \mathrm{a})^{\mathrm{I}} = \mathtt{T}$ since all of the following hold:

$$\mathrm{f}(\mathrm{c})^{\mathrm{I}} = \mathrm{f}^{\mathrm{I}}(1) = 1 \tag{99}$$

$$\mathrm{a}^{\mathrm{I}} = 2 \tag{100}$$

$$\langle 1, 2\rangle \in \mathrm{Q}^{\mathrm{I}}. \tag{101}$$

Exercise 51. *Give an interpretation* I *such that* $\mathrm{Q}(\mathrm{f}(\mathrm{c}), \mathrm{a})^{\mathrm{I}} = \mathtt{F}$.

Note 7. *All we need to do is make one small adjustment to the interpretation in exercise 50.*

To make the formula false, we need to make sure the tuple $\langle 1, 2 \rangle$ *is not in* Q^{I}. *Let* Q^{I} *be the empty set.*

Solution: The modified interpretation I is given below.

- $\mathrm{dom}(\mathrm{I}) = \{1, 2, 3\}$.
- $\mathrm{a}^{\mathrm{I}} = 2, \mathrm{b}^{\mathrm{I}} = 1, \mathrm{c}^{\mathrm{I}} = 1$.
- $\mathrm{f}^{\mathrm{I}}(\mathrm{x}) = \mathrm{x}, \forall \mathrm{x} \in \mathrm{dom}(\mathrm{I})$, $\mathrm{g}^{\mathrm{I}}(\mathrm{x}) = 1, \forall \mathrm{x} \in \mathrm{dom}(\mathrm{I})$.
- $\mathrm{Q}^{\mathrm{I}} = \emptyset$, $\mathrm{P}^{\mathrm{I}} = \emptyset$.

Therefore, $\mathrm{Q}(\mathrm{f}(\mathrm{c}), \mathrm{a})^{\mathrm{I}} = \mathtt{F}$ since all of the following hold:

$$\mathrm{f}(\mathrm{c})^{\mathrm{I}} = \mathrm{f}^{\mathrm{I}}(1) = 1 \tag{102}$$
$$\mathrm{a}^{\mathrm{I}} = 2 \tag{103}$$
$$\langle 1, 2 \rangle \notin \mathrm{Q}^{\mathrm{I}}. \tag{104}$$

2.2.2 Evaluating Formulas with Free Variables Only

Exercise 52. *Give an interpretation* I *and an environment* E *such that* $\mathrm{Q}(\mathrm{f}(\mathrm{x}), \mathrm{a})^{(\mathrm{I},\mathrm{E})} = \mathtt{T}$.

Note 8. *Let's start with the interpretation in the solution to exercise 50. Consider an arbitrary environment* E*. We simplify the formula below.*

$$\mathrm{f}(\mathrm{x})^{(\mathrm{I},\mathrm{E})} = \mathrm{f}^{\mathrm{I}}(\mathrm{E}(\mathrm{x})) = \mathrm{E}(\mathrm{x}) \tag{105}$$

$$\mathrm{a}^{\mathrm{I}} = 2 \tag{106}$$

Thus, the formula is true if and only if $\langle \mathrm{E}(\mathrm{x}), 2 \rangle \in \mathrm{Q}^{\mathrm{I}}$.

The only tuple in Q^{I} *is* $\langle 1, 2 \rangle$*. Thus, it is sufficient to let* $\mathrm{E}(\mathrm{x}) = 1$*. Even though* y *and* z *do not appear in the formula, we still need to include mappings for them in the definition of the environment. Let* $\mathrm{E}(\mathrm{y}) = 1$ *and* $\mathrm{E}(\mathrm{z}) = 1$.

Solution: The interpretation I and the environment E are given below.

- $\mathrm{dom}(\mathrm{I}) = \{1, 2, 3\}$.
- $\mathrm{a}^{\mathrm{I}} = 2, \mathrm{b}^{\mathrm{I}} = 1, \mathrm{c}^{\mathrm{I}} = 1$.
- $\mathrm{f}^{\mathrm{I}}(\mathrm{x}) = \mathrm{x}, \forall \mathrm{x} \in \mathrm{dom}(\mathrm{I})$, $\mathrm{g}^{\mathrm{I}}(\mathrm{x}) = 1, \forall \mathrm{x} \in \mathrm{dom}(\mathrm{I})$.
- $\mathrm{Q}^{\mathrm{I}} = \{\langle 1, 2 \rangle\}$, $\mathrm{P}^{\mathrm{I}} = \emptyset$.

The environment E is $\mathrm{E}(\mathrm{x}) = 1$, $\mathrm{E}(\mathrm{y}) = 1$, and $\mathrm{E}(\mathrm{z}) = 1$.
Given I and E, we can show that $\mathrm{Q}(\mathrm{f}(\mathrm{x}), \mathrm{a})^{(\mathrm{I},\mathrm{E})} = \mathtt{T}$ because

$$\mathrm{E}(\mathrm{x}) = 1 \tag{107}$$

$$\mathrm{f}(\mathrm{x})^{(\mathrm{I},\mathrm{E})} = \mathrm{f}^{\mathrm{I}}(1) = 1 \tag{108}$$

$$\mathrm{a}^{\mathrm{I}} = 2 \tag{109}$$

$$\langle 1, 2 \rangle \in \mathrm{Q}^{\mathrm{I}}. \tag{110}$$

Exercise 53. *Give an interpretation* I *such that* $\mathrm{Q}(\mathrm{f}(\mathrm{x}), \mathrm{a})^{(\mathrm{I},\mathrm{E})} = \mathtt{F}$.

Note 9. *Let's start with the interpretation in the solution to exercise 52, and modify* Q^{I} *to be the empty set.*
Under I *and* E*, the formula is false, using similar reasoning as exercise 51.*

Solution: The interpretation I and the environment E are given below.

- $\mathrm{dom}(\mathrm{I}) = \{1, 2, 3\}$.
- $\mathrm{a}^{\mathrm{I}} = 2, \mathrm{b}^{\mathrm{I}} = 1, \mathrm{c}^{\mathrm{I}} = 1$.
- $\mathrm{f}^{\mathrm{I}}(\mathrm{x}) = \mathrm{x}, \forall \mathrm{x} \in \mathrm{dom}(\mathrm{I})$, $\mathrm{g}^{\mathrm{I}}(\mathrm{x}) = 1, \forall \mathrm{x} \in \mathrm{dom}(\mathrm{I})$.
- $\mathrm{Q}^{\mathrm{I}} = \{\langle 1, 2\rangle\}$, $\mathrm{P}^{\mathrm{I}} = \emptyset$.

The environment E is $\mathrm{E}(\mathrm{x}) = 1$, $\mathrm{E}(\mathrm{y}) = 1$, and $\mathrm{E}(\mathrm{z}) = 1$.

Given I and E, we can show that $\mathrm{Q}(\mathrm{f}(\mathrm{x}), \mathrm{a})^{(\mathrm{I},\mathrm{E})} = \mathtt{F}$ because

$$\mathrm{E}(\mathrm{x}) = 1 \tag{111}$$

$$\mathrm{f}(\mathrm{x})^{(\mathrm{I},\mathrm{E})} = \mathrm{f}^{\mathrm{I}}(1) = 1 \tag{112}$$

$$\mathrm{a}^{\mathrm{I}} = 2 \tag{113}$$

$$\langle 1, 2\rangle \notin \mathrm{Q}^{\mathrm{I}}. \tag{114}$$

2.2.3 Evaluating Formulas with Free and Bound Variables

Exercise 54. *Give an interpretation* I *and an environment* E *such that* $(\exists x\ Q(x,y))^{(I,E)} = \mathtt{T}$. *Assume that the domain is* $\mathrm{dom}(I) = \{1,2,3\}$.

Note 10. *Here is more explanation to help you understand how I came up with the* I *and* E *above.*
y *is a free variable in* α. *The value of* y *is given by the environment. Let's arbitrarily define* $E(y) = 2$. *Even though* x *and* z *do not appear in the formula, we need to define their mappings as part of* E. *Let* $E(x) = 1$ *and* $E(z) = 1$.

To make α *true, there must be at least one tuple in* Q^I *and the second value in the tuple (the value of* y *in the tuple) must be* 2 *because* $E(y) = 2$. *Let* Q^I *be* $\{\langle 1,2\rangle\}$.

We need to define the rest of I *even though the symbols do not appear in the formula.*

Solution: The interpretation I is shown below.

- $\mathrm{dom}(I) = \{1,2,3\}$.
- $a^I = 2, b^I = 1, c^I = 1$.
- $f^I(x) = x, \forall x \in \mathrm{dom}(I)$, $g^I(x) = 1, \forall x \in \mathrm{dom}(I)$.
- $Q^I = \{\langle 1,2\rangle\}$, $P^I = \emptyset$.

The environment E is $E(x) = 1, E(y) = 2, E(z) = 1$.
Given the I and E above, we know that $Q(x,y)^{(I,E[x\mapsto 1])} = \mathtt{T}$ because all of the following hold:

$$E[x \mapsto 1](x) = 1 \tag{115}$$
$$E[x \mapsto 1](y) = 2 \tag{116}$$
$$\langle 1,2\rangle \in Q^I \tag{117}$$

Hence, by the satisfaction rules for $\exists$, $(\exists x\ Q(x,y))^{(I,E)} = \mathtt{T}$.

Exercise 55. *Give an interpretation* I *and an environment* E *such that* $(\forall \mathrm{x}\ \mathrm{Q}(\mathrm{x},\mathrm{y}))^{(\mathrm{I},\mathrm{E})} = \mathtt{T}$. *Assume that the domain is* $\mathrm{dom}(\mathrm{I}) = \{1,2,3\}$.

Note 11. *Let's start with the* I *and* E *in exercise 54.*

We will modify Q^{I}. *To make the formula true, we must be able to replace x by any value in the domain. Furthermore, for each tuple in* Q^{I}, *the second value in the tuple must be* 2 *because the environment maps* y *to* 2. *Thus, let* $\mathrm{Q}^{\mathrm{I}} = \{\langle 1,2\rangle, \langle 2,2\rangle, \langle 3,2\rangle\}$.

Solution: The interpretation I is shown below.

- $\mathrm{dom}(\mathrm{I}) = \{1,2,3\}$.
- $\mathrm{a}^{\mathrm{I}} = 2, \mathrm{b}^{\mathrm{I}} = 1, \mathrm{c}^{\mathrm{I}} = 1$.
- $\mathrm{f}^{\mathrm{I}}(\mathrm{x}) = \mathrm{x}, \forall \mathrm{x} \in \mathrm{dom}(\mathrm{I})$, $\mathrm{g}^{\mathrm{I}}(\mathrm{x}) = 1, \forall \mathrm{x} \in \mathrm{dom}(\mathrm{I})$.
- $\mathrm{Q}^{\mathrm{I}} = \{\langle 1,2\rangle, \langle 2,2\rangle, \langle 3,2\rangle\}$, $\mathrm{P}^{\mathrm{I}} = \emptyset$.

The environment E is $\mathrm{E}(\mathrm{x}) = 1, \mathrm{E}(\mathrm{y}) = 2, \mathrm{E}(\mathrm{z}) = 1$.

We will prove that $(\forall \mathrm{x}\ \mathrm{Q}(\mathrm{x},\mathrm{y}))^{(\mathrm{I},\mathrm{E})} = \mathtt{T}$. Consider all possible values of x. By the definition of Q^{I}, the following statements hold.

- $[\mathrm{x} \mapsto 1]$: $\mathrm{Q}(\mathrm{x},\mathrm{y})^{(\mathrm{I},\mathrm{E}[\mathrm{x}\mapsto 1])} = \mathtt{T}$ because all of the following hold.

$$\mathrm{E}[\mathrm{x} \mapsto 1](\mathrm{x}) = 1 \tag{118}$$
$$\mathrm{E}[\mathrm{x} \mapsto 1](\mathrm{y}) = 2 \tag{119}$$
$$\langle 1,2\rangle \in \mathrm{Q}^{\mathrm{I}}. \tag{120}$$

- $[\mathrm{x} \mapsto 2]$: $\mathrm{Q}(\mathrm{x},\mathrm{y})^{(\mathrm{I},\mathrm{E}[\mathrm{x}\mapsto 2])} = \mathtt{T}$ because all of the following hold.

$$\mathrm{E}[\mathrm{x} \mapsto 2](\mathrm{x}) = 2 \tag{121}$$
$$\mathrm{E}[\mathrm{x} \mapsto 2](\mathrm{y}) = 2 \tag{122}$$
$$\langle 2,2\rangle \in \mathrm{Q}^{\mathrm{I}}. \tag{123}$$

- $[\mathrm{x} \mapsto 3]$: $\mathrm{Q}(\mathrm{x},\mathrm{y})^{(\mathrm{I},\mathrm{E}[\mathrm{x}\mapsto 3])} = \mathtt{T}$ because all of the following hold.

$$\mathrm{E}[\mathrm{x} \mapsto 3](\mathrm{x}) = 3 \tag{124}$$
$$\mathrm{E}[\mathrm{x} \mapsto 3](\mathrm{y}) = 3 \tag{125}$$
$$\langle 3,3\rangle \in \mathrm{Q}^{\mathrm{I}}. \tag{126}$$

Therefore, by the satisfaction rules for $\forall$, $(\forall \mathrm{x}\ \mathrm{Q}(\mathrm{x},\mathrm{y}))^{(\mathrm{I},\mathrm{E})} = \mathtt{T}$.

2.2.4 Evaluating Formulas with Bound Variables Only

Exercise 56. *Give an interpretation* I *and an environment* E *such that* $(\exists x(\forall y\ Q(x,y)))^{(I,E)} = \mathtt{T}$*. Start with the domain* $\mathrm{dom}(I) = \{1,2,3\}$*.*

Note 12. *To make the formula true, there must be at least 3 tuples in* Q^I *because* y *(the second value of each tuple) could take any of the 3 possible values in the domain.*

The first element of all three tuples must be the same because there must be one value for x *that makes* $Q(x,y)$ *true.*

Note that, when choosing the value of x*, we do not know the value of* y *yet. Our choice of value for* x *cannot depend on the value of* y*.*

One definition of Q^I *that satisfies all these requirements is* $Q^I = \{\langle 1,1\rangle, \langle 1,2\rangle, \langle 1,3\rangle\}$*.*

Solution: The interpretation I is given below.

- $\mathrm{dom}(I) = \{1,2,3\}$.
- $a^I = 2, b^I = 1, c^I = 1$.
- $f^I(x) = x, \forall x \in \mathrm{dom}(I)$, $g^I(x) = 1, \forall x \in \mathrm{dom}(I)$.
- $Q^I = \{\langle 1,1\rangle, \langle 1,2\rangle, \langle 1,3\rangle\}$, $P^I = \emptyset$.

Let E be an arbitrary environment.

We will prove that $(\exists x(\forall y\ Q(x,y)))^{(I,E)} = \mathtt{T}$.

By the satisfaction rules of $\exists$, we need to show that $(\forall y\ Q(x,y))^{(I,E[x\mapsto d_x])} = \mathtt{T}$ for some $d_x$indom(I).
Consider $d_x = 1$. We now need to show that $Q(x,y)^{(I,E[x\mapsto d_x][y\mapsto d_y)} = \mathtt{T}$ for every $d_y \in \mathrm{dom}(I)$. Consider all possible values of y.

- $[y \mapsto 1]$: $Q(x,y)^{(I,E[x\mapsto 1][y\mapsto 1])} = \mathtt{T}$ because all of the following hold.

$$E[x \mapsto 1][y \mapsto 1](x) = 1 \tag{127}$$
$$E[x \mapsto 1][y \mapsto 1](y) = 1 \tag{128}$$
$$\langle E[x \mapsto 1][y \mapsto 1](x), E[x \mapsto 1][y \mapsto 1](y)\rangle = \langle 1,1\rangle \in Q^I. \tag{129}$$

- $[y \mapsto 2]$: $Q(x,y)^{(I,E[x\mapsto 1][y\mapsto 2])} = \mathtt{T}$ because all of the following hold.

$$E[x \mapsto 1][y \mapsto 2](x) = 1 \tag{130}$$
$$E[x \mapsto 1][y \mapsto 2](y) = 2 \tag{131}$$
$$\langle E[x \mapsto 1][y \mapsto 2](x), E[x \mapsto 1][y \mapsto 2](y)\rangle = \langle 1,2\rangle \in Q^I. \tag{132}$$

- $[y \mapsto 3]$: $Q(x, y)^{(I, E[x \mapsto 1][y \mapsto 3])} = \mathtt{T}$ because all of the following hold.

$$E[x \mapsto 1][y \mapsto 3](x) = 1 \quad (133)$$

$$E[x \mapsto 1][y \mapsto 3](y) = 3 \quad (134)$$

$$\langle E[x \mapsto 1][y \mapsto 3](x), E[x \mapsto 1][y \mapsto 3](y)\rangle = \langle 1, 3\rangle \in Q^I. \quad (135)$$

By the satisfaction rules of $\forall$, $(\exists x(\forall y\ Q(x, y)))^{(I, E[x \mapsto 1])} = \mathtt{T}$ holds. By the definition of $\exists$, $(\exists x(\forall y\ Q(x, y)))^{(I, E)} = \mathtt{T}$ holds.

Exercise 57. *Give an interpretation* I *and an environment* E *such that* $(\exists x(\forall y\ Q(x,y)))^{(I,E)} = \texttt{F}$*. Start with the domain* $dom(I) = \{1,2,3\}$*.*

Note 13. *The formula has no free variables. The bound variables get their meanings through the quantifiers. Thus, there is no need to define an environment. We only need to define an interpretation to evaluate the formula.*

There are many ways to make the formula false. An easy solution is to let Q^I *be the empty set. Then,* $Q^I(x,y)$ *is always false and the formula must be false as well.*

If there are tuples in Q^I*, we need to make sure that* Q^I *does not have three tuples such that the first value of all three tuples are the same and the second value in all three tuples are all different.*

Solution: The interpretation I is shown below.

- $dom(I) = \{1,2,3\}$.
- $a^I = 2, b^I = 1, c^I = 1$.
- $f^I(x) = x, \forall x \in dom(I)$, $g^I(x) = 1, \forall x \in dom(I)$.
- $Q^I = \{\langle 1,1\rangle, \langle 2,2\rangle, \langle 3,3\rangle\}$, $P^I = \emptyset$.

Let E be an arbitrary environment.

We will prove that $(\exists x(\forall y\ Q(x,y)))^{(I,E)} = \texttt{F}$.

By the satisfaction rules for $\exists$, we need to show that $(\forall y\ Q(x,y))^{(I,E[x\mapsto d_x])} = \texttt{F}$ holds for every $d_x \in dom(I)$.

Consider all possible values of x.

- $[x \mapsto 1]$:

  By the rules of satisfaction for $\forall$, to prove that $(\forall y\ Q(x,y))^{(I,E[x\mapsto 1])} = \texttt{F}$, we need to prove that $Q(x,y)^{(I,E[x\mapsto 1][y\mapsto d_y])} = \texttt{F}$ for some $d_y \in dom(I)$.

  $Q(x,y)^{(I,E[x\mapsto 1][y\mapsto 2])} = \texttt{F}$ holds since all of the following statements hold.

  $$E[x \mapsto 1][y \mapsto 2](x) = 1 \tag{136}$$
  $$E[x \mapsto 1][y \mapsto 2](y) = 2 \tag{137}$$
  $$\langle E[x \mapsto 1][y \mapsto 2](x), E[x \mapsto 1][y \mapsto 2](y)\rangle = \langle 1,2\rangle \notin Q^I \tag{138}$$

  Therefore, $Q(x,y)^{(I,E[x\mapsto 1][y\mapsto 2])} = \texttt{F}$ holds, which means that $(\forall y\ Q(x,y))^{(I,E[x\mapsto 1])} = \texttt{F}$ holds.

- $[\mathrm{x} \mapsto 2]$:
  $\mathrm{Q}(\mathrm{x},\mathrm{y})^{(\mathrm{I},\mathrm{E}[\mathrm{x}\mapsto 2][\mathrm{y}\mapsto 1])} = \mathtt{F}$ holds because all of the following statements hold.

$$\mathrm{E}[\mathrm{x} \mapsto 2][\mathrm{y} \mapsto 1](\mathrm{x}) = 2 \tag{139}$$

$$\mathrm{E}[\mathrm{x} \mapsto 2][\mathrm{y} \mapsto 1](\mathrm{y}) = 1 \tag{140}$$

$$\langle \mathrm{E}[\mathrm{x} \mapsto 2][\mathrm{y} \mapsto 1](\mathrm{x}), \mathrm{E}[\mathrm{x} \mapsto 2][\mathrm{y} \mapsto 1](\mathrm{y}) \rangle = \langle 2, 1 \rangle \notin \mathrm{Q}^{\mathrm{I}} \tag{141}$$

  Therefore, $(\forall \mathrm{y}\ \mathrm{Q}(\mathrm{x},\mathrm{y}))^{(\mathrm{I},\mathrm{E}[\mathrm{x}\mapsto 2])} = \mathtt{F}$ holds.

- $[\mathrm{x} \mapsto 3]$:
  $\mathrm{Q}(\mathrm{x},\mathrm{y})^{(\mathrm{I},\mathrm{E}[\mathrm{x}\mapsto 3][\mathrm{y}\mapsto 1])} = \mathtt{F}$ holds because all of the following statements hold.

$$\mathrm{E}[\mathrm{x} \mapsto 3][\mathrm{y} \mapsto 1](\mathrm{x}) = 3 \tag{142}$$

$$\mathrm{E}[\mathrm{x} \mapsto 3][\mathrm{y} \mapsto 1](\mathrm{y}) = 1 \tag{143}$$

$$\langle \mathrm{E}[\mathrm{x} \mapsto 3][\mathrm{y} \mapsto 1](\mathrm{x}), \mathrm{E}[\mathrm{x} \mapsto 3][\mathrm{y} \mapsto 1](\mathrm{y}) \rangle = \langle 3, 1 \rangle \notin \mathrm{Q}^{\mathrm{I}} \tag{144}$$

  Therefore, $(\forall \mathrm{y}\ \mathrm{Q}(\mathrm{x},\mathrm{y}))^{(\mathrm{I},\mathrm{E}[\mathrm{x}\mapsto 3])} = \mathtt{F}$ holds.

By the satisfaction rules of $\exists$, we have proven that $(\exists \mathrm{x}(\forall \mathrm{y}\ \mathrm{Q}(\mathrm{x},\mathrm{y})))^{(\mathrm{I},\mathrm{E})} = \mathtt{F}$.

2.3 Semantic Entailment

Exercise 58. *Show that* $\{(\forall \mathrm{x}\ \mathrm{P}(\mathrm{x}))\} \models (\exists \mathrm{x}\ \mathrm{P}(\mathrm{x}))$.

Proof. Consider an interpretation I such that $(\forall \mathrm{x}\ \mathrm{P}(\mathrm{x}))^{\mathrm{I}} = \mathtt{T}$. We will prove that $(\exists \mathrm{x}\ \mathrm{P}(\mathrm{x}))^{\mathrm{I}} = \mathtt{T}$.

Consider an arbitrary environment E. Let $\mathrm{d}_1 \in \mathrm{dom}(\mathrm{I})$ be a domain element.

By the satisfaction rules for $\forall$, $\mathrm{P}(\mathrm{x})^{(\mathrm{I},\mathrm{E}[\mathrm{x}\mapsto \mathrm{d}_1])} = \mathtt{T}$. Therefore, $\mathrm{E}[\mathrm{x} \mapsto \mathrm{d}_1](\mathrm{x}) = \mathrm{d}_1 \in \mathrm{P}^{\mathrm{I}}$.

By the satisfaction rules for $\exists$, $(\exists \mathrm{x}\ \mathrm{P}(\mathrm{x}))^{\mathrm{I}} = \mathtt{T}$. □

Exercise 59. *Show that* $\{(\exists \mathrm{x}\ \mathrm{P}(\mathrm{x}))\} \not\models (\forall \mathrm{x}\ \mathrm{P}(\mathrm{x}))$.

Proof. To prove that the entailment does hold, we need to find an interpretation I such that $(\exists \mathrm{x}\ \mathrm{P}(\mathrm{x}))^{\mathrm{I}} = \mathtt{T}$ and $(\forall \mathrm{x}\ \mathrm{P}(\mathrm{x}))^{\mathrm{I}} = \mathtt{F}$.

Consider the interpretation I below.

- $\mathrm{dom}(\mathrm{I}) = \{1, 2\}$.
- $\mathrm{P}^{\mathrm{I}} = \{1\}$.

Let E be an arbitrary environment.

$\mathrm{P}(\mathrm{x})^{(\mathrm{I},\mathrm{E}[\mathrm{x}\mapsto 1])} = \mathtt{T}$ holds since $\mathrm{E}[\mathrm{x} \mapsto 1](\mathrm{x}) = 1 \in \mathrm{P}^{\mathrm{I}}$. By the satisfaction rules for $\exists$, $(\exists \mathrm{x}\ \mathrm{P}(\mathrm{x}))^{(\mathrm{I},\mathrm{E})} = \mathtt{T}$.

$\mathrm{P}(\mathrm{x})^{(\mathrm{I},\mathrm{E}[\mathrm{x}\mapsto 2])} = \mathtt{F}$ holds since $\mathrm{E}[\mathrm{x} \mapsto 2](\mathrm{x}) = 2 \notin \mathrm{P}^{\mathrm{I}}$. By the satisfaction rules for $\forall$, $(\forall \mathrm{x}\ \mathrm{P}(\mathrm{x}))^{(\mathrm{I},\mathrm{E})} = \mathtt{F}$ holds. □

Exercise 60. *Show that* $\{(\exists \mathrm{y}\ (\forall \mathrm{x}\ \mathrm{Q}(\mathrm{x},\mathrm{y})))\} \vDash (\forall \mathrm{x}\ (\exists \mathrm{y}\ \mathrm{Q}(\mathrm{x},\mathrm{y})))$.

Proof. Consider an interpretation I such that $(\exists \mathrm{y}\ (\forall \mathrm{x}\ \mathrm{Q}(\mathrm{x},\mathrm{y})))^{(\mathrm{I},\mathrm{E})} = \mathtt{T}$. We will prove that $(\forall \mathrm{x}\ (\exists \mathrm{y}\ \mathrm{Q}(\mathrm{x},\mathrm{y})))^{(\mathrm{I},\mathrm{E})} = \mathtt{T}$. Let E be an arbitrary environment.

By the satisfaction rules for ∃, we have

$$(\forall \mathrm{x}\ \mathrm{Q}(\mathrm{x},\mathrm{y}))^{(\mathrm{I},\mathrm{E}[\mathrm{y}\mapsto \mathrm{d_y}])} = \mathtt{T} \text{ for some } \mathrm{d_y} \in \mathrm{dom}(\mathrm{I}).$$

By the satisfaction rules for ∀, we have

$$\mathrm{Q}(\mathrm{x},\mathrm{y})^{(\mathrm{I},\mathrm{E}[\mathrm{y}\mapsto \mathrm{d_y}][\mathrm{x}\mapsto \mathrm{d}])} = \mathtt{T}, \text{ for some } \mathrm{d_y} \in \mathrm{dom}(\mathrm{I}) \text{ and for every } \mathrm{d} \in \mathrm{dom}(\mathrm{I}).$$

Note that the environment $\mathrm{E}[\mathrm{y}\mapsto \mathrm{d_y}][\mathrm{x}\mapsto \mathrm{d}]$ is the same environment as $\mathrm{E}[\mathrm{x}\mapsto \mathrm{d}][\mathrm{y}\mapsto \mathrm{d_y}]$. (Their effects on the variables x and y are identical.) Thus, we can re-write the above fact as

$$\mathrm{Q}(\mathrm{x},\mathrm{y})^{(\mathrm{I},\mathrm{E}[\mathrm{x}\mapsto \mathrm{d}][\mathrm{y}\mapsto \mathrm{d_y}])} = \mathtt{T} \text{ for every } \mathrm{d} \in \mathrm{dom}(\mathrm{I}) \text{ and for some } \mathrm{d_y} \in \mathrm{dom}(\mathrm{I}).$$

By the satisfaction rule for ∃, we have

$$(\exists \mathrm{y}\ \mathrm{Q}(\mathrm{x},\mathrm{y}))^{(\mathrm{I},\mathrm{E}[\mathrm{x}\mapsto \mathrm{d}])} = \mathtt{T} \text{ for every } \mathrm{d} \in \mathrm{dom}(\mathrm{I}).$$

By the satisfaction rule for ∀, we have that

$$(\forall \mathrm{x}\ (\exists \mathrm{y}\ \mathrm{Q}(\mathrm{x},\mathrm{y})))^{(\mathrm{I},\mathrm{E})} = \mathtt{T}.$$

□

Exercise 61. *Show that* $\{(\forall \mathrm{x}\ (\exists \mathrm{y}\ \mathrm{Q}(\mathrm{x},\mathrm{y})))\} \not\models (\exists \mathrm{y}\ (\forall \mathrm{x}\ \mathrm{Q}(\mathrm{x},\mathrm{y})))$.

Proof. To prove that the entailment does not hold, we need to find an interpretation I such that $(\forall \mathrm{x}\ (\exists \mathrm{y}\ \mathrm{Q}(\mathrm{x},\mathrm{y})))^{(\mathrm{I},\mathrm{E})} = \mathtt{T}$ and $(\exists \mathrm{y}\ (\forall \mathrm{x}\ \mathrm{Q}(\mathrm{x},\mathrm{y})))^{(\mathrm{I},\mathrm{E})} = \mathtt{F}$.

Consider the interpretation I below.

- $\mathrm{dom}(\mathrm{I}) = \{1, 2\}$.
- $\mathrm{Q}^{\mathrm{I}} = \{\langle 1, 1\rangle, \langle 2, 2\rangle\}$.

First, we will show that $(\forall \mathrm{x}\ (\exists \mathrm{y}\ \mathrm{Q}(\mathrm{x},\mathrm{y})))^{(\mathrm{I},\mathrm{E})} = \mathtt{T}$. Let E be an arbitrary environment. Consider all possible values of x.

- $[\mathrm{x} \mapsto 1]$: $\mathrm{Q}(\mathrm{x},\mathrm{y})^{(\mathrm{I},\mathrm{E}[\mathrm{x}\mapsto 1][\mathrm{y}\mapsto 1])} = \mathtt{T}$ because

  $$\langle \mathrm{E}[\mathrm{x} \mapsto 1][\mathrm{y} \mapsto 1](\mathrm{x}), \mathrm{E}[\mathrm{x} \mapsto 1][\mathrm{y} \mapsto 1](\mathrm{y})\rangle = \langle 1, 1\rangle \in \mathrm{Q}^{\mathrm{I}}.$$

  By the satisfaction rule for $\exists$, $(\exists \mathrm{y}\ \mathrm{Q}(\mathrm{x},\mathrm{y}))^{(\mathrm{I},\mathrm{E}[\mathrm{x}\mapsto 1])} = \mathtt{T}$.

- $[\mathrm{x} \mapsto 2]$: $\mathrm{Q}(\mathrm{x},\mathrm{y})^{(\mathrm{I},\mathrm{E}[\mathrm{x}\mapsto 2][\mathrm{y}\mapsto 2])} = \mathtt{T}$ because

  $$\langle \mathrm{E}[\mathrm{x} \mapsto 2][\mathrm{y} \mapsto 2](\mathrm{x}), \mathrm{E}[\mathrm{x} \mapsto 2][\mathrm{y} \mapsto 2](\mathrm{y})\rangle = \langle 2, 2\rangle \in \mathrm{Q}^{\mathrm{I}}.$$

  By the satisfaction rule for $\exists$, $(\exists \mathrm{y}\ \mathrm{Q}(\mathrm{x},\mathrm{y}))^{(\mathrm{I},\mathrm{E}[\mathrm{x}\mapsto 2])} = \mathtt{T}$.

Thus, by the satisfaction rule for $\forall$, $(\forall \mathrm{x}\ (\exists \mathrm{y}\ \mathrm{Q}(\mathrm{x},\mathrm{y})))^{(\mathrm{I},\mathrm{E})} = \mathtt{T}$.

Next, we will show that $(\exists \mathrm{y}\ (\forall \mathrm{x}\ \mathrm{Q}(\mathrm{x},\mathrm{y})))^{(\mathrm{I},\mathrm{E})} = \mathtt{F}$. Let E be an arbitrary environment. Consider all possible values of y.

- $[\mathrm{y} \mapsto 1]$: $\mathrm{Q}(\mathrm{x},\mathrm{y})^{(\mathrm{I},\mathrm{E}[\mathrm{x}\mapsto 2][\mathrm{y}\mapsto 1])} = \mathtt{F}$ because

  $$\langle \mathrm{E}[\mathrm{x} \mapsto 2][\mathrm{y} \mapsto 1](\mathrm{x}), \mathrm{E}[\mathrm{x} \mapsto 2][\mathrm{y} \mapsto 1](\mathrm{y})\rangle = \langle 2, 1\rangle \notin \mathrm{Q}^{\mathrm{I}}.$$

  By the satisfaction rule for $\exists$, $(\forall \mathrm{x}\ \mathrm{Q}(\mathrm{x},\mathrm{y}))^{(\mathrm{I},\mathrm{E}[\mathrm{y}\mapsto 1])} = \mathtt{F}$.

- $[\mathrm{y} \mapsto 2]$: $\mathrm{Q}(\mathrm{x},\mathrm{y})^{(\mathrm{I},\mathrm{E}[\mathrm{x}\mapsto 1][\mathrm{y}\mapsto 2])} = \mathtt{F}$ because

  $$\langle \mathrm{E}[\mathrm{x} \mapsto 1][\mathrm{y} \mapsto 2](\mathrm{x}), \mathrm{E}[\mathrm{x} \mapsto 1][\mathrm{y} \mapsto 2](\mathrm{y})\rangle = \langle 1, 2\rangle \notin \mathrm{Q}^{\mathrm{I}}.$$

  By the satisfaction rule for $\forall$, $(\forall \mathrm{x}\ \mathrm{Q}(\mathrm{x},\mathrm{y}))^{(\mathrm{I},\mathrm{E}[\mathrm{y}\mapsto 2])} = \mathtt{F}$.

Thus, by the satisfaction rule for $\forall$, $(\exists \mathrm{y}\ (\forall \mathrm{x}\ \mathrm{Q}(\mathrm{x},\mathrm{y})))^{(\mathrm{I},\mathrm{E})} = \mathtt{F}$.

Hence, the entailment does not hold. □

Exercise 62. *Show that* $\{(\forall x\ (\alpha \to \beta))\} \vDash ((\forall x\ \alpha) \to (\forall x\ \beta))$*, where* x *is a variable symbol and* α *and* β *are well-formed predicate formulas.*

Proof. Consider an interpretation I and an environment E such that $(\forall x\ (\alpha \to \beta))^{(I,E)} = \mathtt{T}$. We will prove that $((\forall x\ \alpha) \to (\forall x\ \beta))^{(I,E)} = \mathtt{T}$.

To show that $((\forall x\ \alpha) \to (\forall x\ \beta))^{(I,E)} = \mathtt{T}$, we assume that $(\forall x\ \alpha)^{(I,E)} = \mathtt{T}$.

By the satisfaction rule for $\forall$, we have that

$$\alpha^{(I,E[x \mapsto d])} = \mathtt{T} \text{ for every } d \in \mathrm{dom}(I).$$

By our assumption, $(\forall x\ (\alpha \to \beta))^{(I,E)} = \mathtt{T}$. By the satisfaction rule for $\forall$, we have that

$$(\alpha \to \beta)^{(I,E[x \mapsto d])} = \mathtt{T} \text{ for every } d \in \mathrm{dom}(I).$$

By the satisfaction rule for an implication, we have that

$$\beta^{(I,E[x \mapsto d])} = \mathtt{T} \text{ for every } d \in \mathrm{dom}(I).$$

By the satisfaction rule for $\forall$, we have that

$$(\forall x\ \beta)^{(I,E)} = \mathtt{T}.$$

Thus, the entailment holds. □

Exercise 63. *Show that* $\{((\forall x\ \alpha) \to (\forall x\ \beta))\} \not\models (\forall x\ (\alpha \to \beta))$*, where* x *is a variable symbol and* α *and* β *are well-formed predicate formulas.*

Note 14. *The most important step for the proof below is to come up with the concrete example such that the premises are all true and the conclusion is false.*

I first chose concrete formulas for α *and* β*. This step is important. Without doing so, I may not be able to make claims about whether* α *and* β *are true or false under a particular interpretation.*

Next, I construct an interpretation to satisfy the two requirements. I start by picking a domain containing two elements. It is small enough to be manageable and large enough to give me a few possibilities to experiment with.

Then, I try to find definitions for P^I *and* Q^I *to satisfy the two requirements.*

First, I want to make the conclusion $(\forall x\ (P(x) \to Q(x)))$ *false. To do this, it is sufficient to make* P *to be true and* Q *to be false for one value of* x *(so that the implication* $(P(x) \to Q(x))$ *is false). I used* $x = 2$ *for this case and made sure that* $2 \in P^I$ *and* $2 \notin Q^I$*.*

Next, I want to make the premise true. Since $2 \notin Q^I$*, then* $(\forall x\ Q(x))$ *is false. So the conclusion of the premise is false. To make the premise true, I have to make the premise of the premise false. This means that, I need to make sure at least one domain element is not in* P^I*. Therefore, I defined* P^I *such that* $1 \notin P^I$*.*

Solution:
Let α be P(x) and let β be Q(x), where P and Q are unary predicates. Consider the following interpretation:

- $\text{dom}(I) = \{1, 2\}$
- $P^I = \{2\}$ and $Q^I = \{1\}$

We need to show that $((\forall x\ P(x)) \to (\forall x\ Q(x)))^{(I,E)} = \mathtt{T}$ and $(\forall x\ (P(x) \to Q(x)))^{(I,E)} = \mathtt{F}$. Let E be an arbitrary environment.

First, we will show that $((\forall x\ P(x)) \to (\forall x\ Q(x)))^{(I,E)} = \mathtt{T}$.

$P(x)^{(I,E[x \mapsto 1])} = \mathtt{F}$ because $E[x \mapsto 1](x) = 1 \notin P^I$. By the satisfaction rule for $\forall$, $(\forall x\ P(x))^{(I,E)} = \mathtt{F}$.

By the satisfaction rule for an implication, $((\forall x\ P(x)) \to (\forall x\ Q(x)))^{(I,E)} = \mathtt{T}$ because $(\forall x\ P(x))^{(I,E)} = \mathtt{F}$.

Next, we will show that $(\forall x\ (P(x) \to Q(x)))^{(I,E)} = \mathtt{F}$.

$(P(x) \to Q(x))^{(I,E[x \mapsto 2])} = \mathtt{F}$ because $E[x \mapsto 2](x) = 2 \in P^I$ and $E[x \mapsto 2](x) = 2 \notin Q^I$.

By the satisfaction rule for $\forall$, $(\forall x\ (P(x) \to Q(x)))^{(I,E)} = \mathtt{F}$.

In summary, the entailment does not hold.

2.4 Natural Deduction

- $\forall$e (analogous to $\wedge$e)
- $\forall$i (analogous to $\wedge$i)
  - We know nothing about the fresh variable u except that u is a domain element. (If u is special, our conclusion may not be valid.)
  - The fresh variable u cannot escape the subproof box. For example, we cannot conclude $\alpha[u/x]$ outside of the box.
  - When you choose the fresh variable u, make sure that it has not appears anywhere outside of the subproof box in the proof.
- $\exists$e (analogous to $\vee$e)
  - Proof by cases.
  - The conclusion may have nothing to do with the starting formula.
- $\exists$i (analogous to $\vee$i)

2.4.1 Forall-elimination

Exercise 64. *Show that* $\{P(t), (\forall x\ (P(x) \to (\neg Q(x))))\} \vdash (\neg Q(t))$.

1.	$P(t)$	premise
2.	$(\forall x\ (P(x) \to (\neg Q(x))))$	premise
3.	$(P(t) \to (\neg Q(t)))$	$\forall$e: 2
4.	$(\neg Q(t))$	$\to$e: 1, 3

2.4.2 Exists-introduction

Exercise 65. *Show that* $\{(\neg P(y))\} \vdash (\exists x\ (P(x) \to Q(y)))$.

1.	$(\neg P(y))$	premise
2.	$P(y)$	assumption
3.	$\perp$	$\perp$i: 1, 2
4.	$Q(y)$	$\perp$e: 3
5.	$(P(y) \to Q(y))$	$\to$i: 2-4
6.	$(\exists x\ (P(x) \to Q(y)))$	$\exists$i: 5

Exercise 66. *Show that* $\{(\forall x\ P(x))\} \vdash (\exists y\ P(y))$.

1.	$(\forall x\ P(x))$	premise
2.	$P(u)$	$\forall$e: 1
3.	$(\exists y\ P(y))$	$\exists$i: 2

2.4.3 Forall-introduction

Exercise 67. *Show that* $\{(\forall x\ P(x))\} \vdash (\forall y\ P(y))$.

1.	$(\forall x\ P(x))$	premise
2.	u fresh	assumption
3.	$P(u)$	$\forall$e: 1
4.	$(\forall y\ P(y))$	$\forall$i: 2-3

Exercise 68. *Show that* $\{(\forall x\ (P(x) \to Q(x))), (\forall x\ P(x))\} \vdash (\forall x\ Q(x))$.

1.	$(\forall x\ (P(x) \to Q(x)))$	premise
2.	$(\forall x\ P(x))$	premise
3.	u fresh	assumption
4.	$P(u)$	$\forall$e: 2
5.	$(P(u) \to Q(u))$	$\forall$e: 1
6.	$Q(u)$	$\to$e: 4, 5
7.	$(\forall x\ Q(x))$	$\forall$i: 3-6

Exercise 69. *Show that* $\{(\forall x\ (P(x) \to Q(x)))\} \vdash ((\forall x\ P(x)) \to (\forall y\ Q(y)))$.

1.	$(\forall x\ (P(x) \to Q(x)))$	premise
2.	$(\forall x\ P(x))$	assumption
3.	u fresh	assumption
4.	$P(u)$	$\forall$e: 2
5.	$(P(u) \to Q(u))$	$\forall$e: 1
6.	$Q(u)$	$\to$e: 4, 5
7.	$((\forall y\ Q(y))$	$\forall$i: 3-6
8.	$((\forall x\ P(x)) \to (\forall y\ Q(y)))$	$\to$i: 2-7

2.4.4 Exists-elimination

Exercise 70. *Show that* $\{(\exists x\ P(x))\} \vdash (\exists y\ P(y))$.

1.	$(\exists x\ P(x))$	premise
2.	$P(u)$, u fresh	assumption
3.	$(\exists y\ P(u))$	∃i: 2
4.	$(\exists y\ P(y))$	∃e: 1, 2-3

Exercise 71. *Show that* $\{(\forall x\ (P(x) \rightarrow Q(x))), (\exists x\ P(x))\} \vdash (\exists x\ Q(x))$.

1.	$(\forall x\ (P(x) \rightarrow Q(x)))$	premise
2.	$(\exists x\ P(x))$	premise
3.	$P(u)$, u fresh	assumption
4.	$(P(u) \rightarrow Q(u))$	∀e: 1
5.	$Q(u)$	→e: 3, 4
6.	$(\exists x\ Q(x))$	∃i: 5
7.	$(\exists x\ Q(x))$	∃e: 2, 3-6

Exercise 72. *Show that* $\{(\forall x\ (Q(x) \rightarrow R(x))), (\exists x\ (P(x) \land Q(x)))\} \vdash (\exists x\ (P(x) \land R(x)))$.

1.	$(\forall x\ (Q(x) \rightarrow R(x)))$	premise
2.	$(\exists x\ (P(x) \land Q(x)))$	premise
3.	$(P(u) \land Q(u))$, u fresh	assumption
4.	$P(u)$	∧e: 3
5.	$Q(u)$	∧e: 3
6.	$(Q(u) \rightarrow R(u))$	∀e: 1
7.	$R(u)$	→e: 5, 6
8.	$(P(u) \land R(u))$	∧i: 4, 7
9.	$(\exists x\ (P(x) \land R(x)))$	∃i: 8
10.	$(\exists x\ (P(x) \land R(x)))$	∃e: 2, 3-9

2.4.5 Putting them together

Exercise 73. *Show that* $\{(\exists x\ P(x)), (\forall x\ (\forall y\ (P(x) \rightarrow Q(y))))\} \vdash (\forall y\ Q(y))$.

1.	$(\exists x\ P(x))$	premise
2.	$(\forall x\ (\forall y\ (P(x) \rightarrow Q(y))))$	premise
3.	y_0 fresh	assumption
4.	$P(x_0)$, x_0 fresh	assumption
5.	$(\forall y\ (P(x_0) \rightarrow Q(y)))$	$\forall$e: 2
6.	$(P(x_0) \rightarrow Q(y_0))$	$\forall$e: 5
7.	$Q(y_0)$	$\rightarrow$e: 4, 6
8.	$Q(y_0)$	$\exists$e: 1, 4-7
9.	$(\forall y\ Q(y))$	$\forall$i: 3-8

Exercise 74. *Show that* $\{(\exists y\ (\forall x\ P(x, y)))\} \vdash (\forall x\ (\exists y\ P(x, y)))$.

1.	$(\exists y\ (\forall x\ P(x, y)))$	premise
2.	$(\forall x\ P(x, y_0))$, y_0 fresh	assumption
3.	x_0 fresh	assumption
4.	$P(x_0, y_0)$	$\forall$e: 2
5.	$(\exists y\ P(x_0, y))$	$\exists$i: 4
6.	$(\forall x\ (\exists y\ P(x, y)))$	$\forall$i: 3-5
7.	$(\forall x\ (\exists y\ P(x, y)))$	$\exists$e: 1, 2-6

Exercise 75. *Show that* $\{(\neg(\exists x\ P(x)))\} \vdash (\forall x\ (\neg P(x)))$. *(De Morgan)*

1.	$(\neg(\exists x\ P(x))$	premise
2.	u fresh	assumption
3.	$P(u)$	assumption
4.	$(\exists x\ P(x))$	$\exists$i: 3
5.	$\bot$	$\bot$i: 1, 4
6.	$(\neg P(u))$	$\neg$i: 3-5
7.	$(\forall x\ (\neg P(x)))$	$\forall$i: 2-6

Exercise 76. *Show that* $\{(\forall x\ (\neg P(x)))\} \vdash (\neg(\exists x\ P(x)))$. *(De Morgan)*

1.	$(\forall x\ (\neg P(x)))$	premise
2.	$(\exists x\ P(x))$	assumption
3.	$P(u)$, u fresh	assumption
4.	$(\neg P(u))$	$\forall$e: 2
5.	$\bot$	$\bot$i: 3, 4
6.	$\bot$	$\exists$e: 3-5
7.	$(\neg(\exists x\ P(x)))$	$\neg$i: 2-6

Exercise 77. *Show that* $\{(\exists x\ (\neg P(x)))\} \vdash (\neg(\forall x\ P(x)))$*. (De Morgan)*

1.	$(\exists x\ (\neg P(x)))$	premise
2.	$(\forall x\ P(x))$	assumption
3.	$(\neg P(u))$, u fresh	assumption
4.	$P(u)$	$\forall$e: 2
5.	$\bot$	$\bot$i: 3, 4
6.	$\bot$	$\exists$e: 3-5
7.	$(\neg(\forall x\ P(x)))$	$\neg$i: 2-6

Exercise 78. *Show that* $\{(\neg(\forall x\ P(x)))\} \vdash (\exists x\ (\neg P(x)))$*. (De Morgan)*

1.	$(\neg(\forall x\ P(x)))$	premise
2.	$(\neg(\exists x\ (\neg P(x))))$	assumption
3.	u fresh	assumption
4.	$(\neg P(u))$	assumption
5.	$(\exists x\ (\neg P(x)))$	$\exists$i: 4
6.	$\bot$	2, 5
7.	$P(u)$	PBC: 4-6
8.	$(\forall x\ P(x))$	$\forall$i: 3-7
9.	$\bot$	$\bot$i: 1, 8
10.	$(\exists x\ (\neg P(x)))$	PBC: 2-9

2.5 Soundness and Completeness of Natural Deduction

2.5.1 Proving that an inference rule is sound or not sound

Lemma 1. *Let* t *be a predicate term. Let* I *be an interpretation with domain* $\mathrm{dom}(\mathrm{I})$*. Let* E *be an environment. Then we have that* $\mathrm{t}^{(\mathrm{I},\mathrm{E})} \in \mathrm{dom}(\mathrm{I})$*.*

Lemma 2. *Let* α *be a well-formed predicate formula. Let* t *be a predicate term. Let* I *and* E *be an interpretation and environment. Let* x *be a variable. Then we have that* $\alpha[\mathrm{t}/\mathrm{x}]^{(\mathrm{I},\mathrm{E})} = \alpha^{(\mathrm{I},\mathrm{E}[\mathrm{x}\mapsto \mathrm{t}^{(\mathrm{I},\mathrm{E})}])}$*.*

Exercise 79. *Prove that the* $\forall e$ *inference rule is sound. That is, prove that the entailment holds:*

$$\{(\forall \mathrm{x}\ \alpha)\} \vDash \alpha[\mathrm{t}/\mathrm{x}] \tag{145}$$

where α *be a Predicate formula,* x *is a variable, and* t *is a Predicate term.*

Proof. Let (I,E) be an interpretation and environment such that $(\forall \mathrm{x}\ \alpha)^{(\mathrm{I},\mathrm{E})} = \mathtt{T}$.

By the satisfaction rule for $\forall$, we have that $\alpha^{(\mathrm{I},\mathrm{E}[\mathrm{x}\mapsto \mathrm{d}])} = \mathtt{T}$, for every $\mathrm{d} \in \mathrm{dom}(\mathrm{I})$.

By Lemma 1, $\mathrm{t}^{(\mathrm{I},\mathrm{E})}$ is some domain element. Thus, we have that $\alpha^{(\mathrm{I},\mathrm{E}[\mathrm{x}\mapsto \mathrm{t}^{(\mathrm{I},\mathrm{E})}])} = \mathtt{T}$.

By Lemma 2, we have that $\alpha[\mathrm{t}/\mathrm{x}]^{(\mathrm{I},\mathrm{E})} = \alpha^{(\mathrm{I},\mathrm{E}[\mathrm{x}\mapsto \mathrm{t}^{(\mathrm{I},\mathrm{E})}])}$. Thus, we have that $\alpha[\mathrm{t}/\mathrm{x}]^{(\mathrm{I},\mathrm{E})} = \mathtt{T}$.

□

Exercise 80. *Prove that the* $\exists i$ *inference rule is sound. That is, prove that the entailment holds:*

$$\{\alpha[\mathrm{t}/\mathrm{x}]\} \vDash (\exists \mathrm{x}\ \alpha) \tag{146}$$

where α *is a predicate formula,* t *is a predicate term, and* x *is a variable.*

Proof. Let (I,E) be an interpretation and environment such that $\alpha[\mathrm{t}/\mathrm{x}]^{(\mathrm{I},\mathrm{E})} = \mathtt{T}$.

By Lemma 2, we have that $\alpha[\mathrm{t}/\mathrm{x}]^{(\mathrm{I},\mathrm{E})} = \alpha^{(\mathrm{I},\mathrm{E}[\mathrm{x}\mapsto \mathrm{t}^{(\mathrm{I},\mathrm{E})}])}$. Thus, we have that $\alpha^{(\mathrm{I},\mathrm{E}[\mathrm{x}\mapsto \mathrm{t}^{(\mathrm{I},\mathrm{E})}])} = \mathtt{T}$.

By Lemma 1, $\mathrm{t}^{(\mathrm{I},\mathrm{E})}$ is some domain element. Thus, by the satisfaction rule for $\exists$, we have that $(\exists \mathrm{x}\ \alpha)^{(\mathrm{I},\mathrm{E})} = \mathtt{T}$.

□

Exercise 81. *Prove that the following inference rule is NOT sound.*

$$\frac{\alpha[\mathrm{t}/\mathrm{x}]}{(\forall \mathrm{x}\ \alpha)}\ \forall \mathrm{i}* \tag{147}$$

where α *is a predicate formula,* t *is a predicate term, and* x *is a variable.*

Proof. We need to provide an interpretation I and an environment E such that $\alpha[\mathrm{t}/\mathrm{x}]^{(\mathrm{I},\mathrm{E})} = \mathtt{T}$ and $(\forall \mathrm{x}\ \alpha)^{(\mathrm{I},\mathrm{E})} = \mathtt{F}$.

Consider the language of predicate logic where $\mathrm{P}^{(1)}$ is a unary predicate and x and y are variables.

Let α be P(x) and let t be y. Let the interpretation I be defined below.

- $\mathrm{dom}(\mathrm{I}) = \{1, 2\}$
- $\mathrm{P}^{\mathrm{I}} = \{1\}$

Let the environment E be defined by $\mathrm{E}(\mathrm{x}) = 1$ and $\mathrm{E}(\mathrm{y}) = 1$.

First, we show that $\alpha[\mathrm{t}/\mathrm{x}]^{(\mathrm{I},\mathrm{E})} = \mathtt{T}$. By Lemma 2, $\alpha[\mathrm{t}/\mathrm{x}]^{(\mathrm{I},\mathrm{E})} = \alpha^{(\mathrm{I},\mathrm{E}[\mathrm{x}\mapsto \mathrm{t}^{(\mathrm{I},\mathrm{E})}])}$. By the definition of the term t, $\mathrm{t}^{(\mathrm{I},\mathrm{E})} = \mathrm{y}^{(\mathrm{I},\mathrm{E})} = \mathrm{E}(\mathrm{y}) = 1$. Thus, $\alpha[\mathrm{t}/\mathrm{x}]^{(\mathrm{I},\mathrm{E})} = \alpha^{(\mathrm{I},\mathrm{E}[\mathrm{x}\mapsto 1])} = \mathrm{P}(\mathrm{x})^{(\mathrm{I},\mathrm{E}[\mathrm{x}\mapsto 1])} = \mathtt{T}$ because $\mathrm{E}[\mathrm{x} \mapsto 1](\mathrm{x}) = 1 \in \mathrm{P}^{\mathrm{I}}$.

Next, we show that $(\forall \mathrm{x}\ \alpha)^{(\mathrm{I},\mathrm{E})} = \mathtt{F}$. By the satisfaction rule for $\forall$, we need to show that $\alpha^{(\mathrm{I},\mathrm{E}[\mathrm{x}\mapsto \mathrm{d}])} = \mathtt{F}$ for at least one $\mathrm{d} \in \mathrm{dom}(\mathrm{I})$. We have that $\alpha^{(\mathrm{I},\mathrm{E}[\mathrm{x}\mapsto 2])} = \mathrm{P}(\mathrm{x})^{(\mathrm{I},\mathrm{E}[\mathrm{x}\mapsto 2])} = \mathtt{F}$ because $\mathrm{E}[\mathrm{x} \mapsto 2](\mathrm{x}) = 2 \notin \mathrm{P}^{\mathrm{I}}$. □

Exercise 82. *Prove that the following inference rule is NOT sound.*

$$\frac{(\forall x(\alpha \to \beta)) \quad \beta[t/x]}{\alpha[t/x]}\ \forall e* \tag{148}$$

where α *and* β *are predicate formulas,* t *is a predicate term, and* x *is a variable.*

Proof. We need to provide an interpretation I and an environment E such that $(\forall x(\alpha \to \beta))^{(I,E)} = \mathtt{T}$, $\beta[t/x]^{(I,E)} = \mathtt{T}$ and $\alpha[t/x]^{(I,E)} = \mathtt{F}$.

Consider the language of predicate logic where $P^{(1)}$ and $Q^{(1)}$ are unary predicates and x and y are variables.

Let α be $(P(x) \wedge Q(x))$. Let β be $P(x)$ and let t be y. Let the interpretation I be defined below.

- $dom(I) = \{1\}$
- $P^I = \{1\}$
- $Q^I = \emptyset$

Let the environment E be defined by $E(x) = 1$ and $E(y) = 1$.

First, we show that $(\forall x(\alpha \to \beta))^{(I,E)} = \mathtt{T}$**.** The domain in I only has 1 element. Consider $x \mapsto 1$.
By the definitions of α and β, we have that

$$(\alpha \to \beta)^{(I,E[x\mapsto 1])} = ((P(x) \wedge Q(x)) \to P(x))^{(I,E[x\mapsto 1])}.$$

By the definition of P, we have that

$$P(x))^{(I,E[x\mapsto 1])} = \mathtt{T}$$

because $E[x \mapsto 1](x) = 1 \in P^I$.
By the satisfaction rule for $\to$, since $P(x))^{(I,E[x\mapsto 1])} = \mathtt{T}$, we have that

$$((P(x) \wedge Q(x)) \to P(x))^{(I,E[x\mapsto 1])} = \mathtt{T}.$$

Thus, we can conclude that

$$(\alpha \to \beta)^{(I,E[x\mapsto 1])} = \mathtt{T}.$$

By the satisfaction rule for $\forall$, we have that

$$(\forall x(\alpha \to \beta))^{(I,E)} = \mathtt{T}.$$

Next, we show that $\beta[t/x]^{(I,E)} = \mathtt{T}$. By Lemma 2, we have that

$$\beta[t/x]^{(I,E)} = \beta^{(I,E[x \mapsto t^{(I,E)}])} = \mathtt{T}.$$

By the definition of the term t, we have that $t^{(I,E)} = y^{(I,E)} = E(y) = 1$. Thus,

$$\beta[t/x]^{(I,E)} = \beta^{(I,E[x \mapsto 1])} = P(x)^{(I,E[x \mapsto 1])} = \mathtt{T}$$

because $E[x \mapsto 1](x) = 1 \in P^I$.

Finally, we show that $\alpha[t/x]^{(I,E)} = \mathtt{F}$.
By Lemma 2 and by the definition of α, we have that

$$\alpha[t/x]^{(I,E)} = \alpha^{(I,E[x \mapsto t^{(I,E)}])} = (P(x) \wedge Q(x))^{(I,E[x \mapsto t^{(I,E)}])}.$$

By the definition of the term t, we have that

$$t^{(I,E)} = y^{(I,E)} = E(y) = 1.$$

By the definition of Q, we have that

$$Q(x))^{(I,E[x \mapsto 1])} = \mathtt{F}$$

because $E[x \mapsto 1](x) = 1 \notin Q^I$.
Thus, by the definition of $\wedge$, we have that

$$(P(x) \wedge Q(x))^{(I,E[x \mapsto t^{(I,E)}])} = \mathtt{F},$$

which is equivalent to $\alpha[t/x]^{(I,E)} = \mathtt{F}$. □

2.5.2 Proofs using the soundness and completeness theorems

Exercise 83. *Show that there is no natural deduction proof for* $\{(\exists \mathrm{x}\ \mathrm{P}(\mathrm{x}))\} \vdash \mathrm{P}(\mathrm{t})$*, where* P *is a unary predicate,* t *is a term and* x *is a variable.*