CS245 Logic and Computation

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Contents

1 Propositional Logic

1.1 Translations

Exercise 1. *Translate the following three sentences into propositional logic.*

- **Nadhi will eat a fruit if it is an apple.**

- **Nadhi will eat a fruit only if it is an apple.**

- **Nadhi will eat a fruit if and only if it is an apple.**

Exercise 2. *Translate the following sentence into multiple propositional formulas. Show that they are logically equivalent using a truth table.*

Soo-Jin will eat an apple or an orange but not both.

Exercise 3. *Translate the following sentence into at least three syntactically different propositional formulas. Show that they are logically equivalent using a truth table.*

If it is sunny tomorrow, then I will play golf, provided that I am relaxed.

Exercise 4. *Translate the following sentence into a propositional formula.*

If I ace CS 245, I will get a job at Google; otherwise I will apply for the Geek Squad.

Exercise 5. *Translate the following sentence into two propositional formulas and prove that the formulas are not logically equivalent using a truth table.*

Sidney will carry an umbrella unless it is sunny.

1.2 Structural Induction

Theorem 1. *Every well-formed formula has an equal number of opening and closing brackets.*

Proof by Structural Induction. Let P(φ) denote that the well-formed formula φ has an equal number of opening and closing brackets. Let op(φ) and cl(φ) denote the number of opening and closing brackets of φ respectively.

Base case: φ is a propositional variable q. Prove that P(q) holds.

Induction step:

Case 1: φ is (¬a), where a is well-formed.
Induction hypothesis: Assume that P(a) holds.
We need to prove that P((¬a)) holds.

Case 2: φ is (a ∗ b) where a and b are well-formed and ∗ is a binary connective.
Induction hypothesis: Assume that P(a) and P(b) hold.
We need to prove that P((a ∗ b)) holds.

By the principle of structural induction, P(φ) holds for every well-formed formula φ.
QED

□

Theorem 2. *Every proper prefix of a well-formed formula has more opening than closing brackets.*

Proof by Structural Induction. Let P(φ) denote that every proper prefix of the well-formed formula φ has more opening than closing brackets.

Base case: φ is a propositional variable q. Prove that P(q) holds.

Induction step:

Case 1: φ is (¬a), where a is well-formed.
Induction hypothesis: Assume that P(a) holds.
We need to prove that P((¬a)) holds.

Case 2: φ is (a ∗ b) where a and b are well-formed and ∗ is a binary connective.
Induction hypothesis: Assume that P(a) and P(b) hold.
We need to prove that P((a ∗ b)) holds.

By the principle of structural induction, P(φ) holds for every well-formed formula φ. QED

□

1.3 Tautology, Contradiction, and Satisfiable but Not a Tautology

Exercise 6. *Determine whether each of the following formulas is a tautology, satisfiable but not a tautology, or a contradiction.*

- p

- $((\mathrm{r} \wedge \mathrm{s}) \rightarrow \mathrm{r})$

- $((\neg(\mathrm{p} \leftrightarrow \mathrm{q})) \leftrightarrow (\mathrm{q} \vee \mathrm{p}))$

- $((((\mathrm{p} \vee \mathrm{q}) \wedge (\mathrm{p} \vee (\neg \mathrm{q}))) \wedge ((\neg \mathrm{p}) \vee \mathrm{q})) \wedge ((\neg \mathrm{p}) \vee (\neg \mathrm{q})))$

1.4 Logical Equivalence

Exercise 7. *"If it is sunny, I will play golf, provided that I am relaxed."*
s*: it is sunny.* g*: I will play golf.* r*: I am relaxed.*

There are three possible translations:

1. $(\mathrm{r} \to (\mathrm{s} \to \mathrm{g}))$
2. $((\mathrm{s} \land \mathrm{r}) \to \mathrm{g})$
3. $(\mathrm{s} \to (\mathrm{r} \to \mathrm{g}))$

Prove that all three translations are logically equivalent.

Problem 1: $(\mathrm{r} \to (\mathrm{s} \to \mathrm{g})) \equiv ((\mathrm{s} \land \mathrm{r}) \to \mathrm{g})$.

Problem 2: $(\mathrm{r} \to (\mathrm{s} \to \mathrm{g})) \equiv (\mathrm{s} \to (\mathrm{r} \to \mathrm{g}))$.

Exercise 8. *”If it snows then I will not go to class but I will do my assignment.”*
s*: it snows.* c*: I will go to class.* a*: I will do my assignment.*

There are two possible translations:

1. $((\mathrm{s} \rightarrow (\neg \mathrm{c})) \wedge \mathrm{a})$
2. $(\mathrm{s} \rightarrow ((\neg \mathrm{c}) \wedge \mathrm{a}))$

Prove that the two translations are NOT logically equivalent.

1.5 Analyzing Conditional Code

Consider the following code fragment:

```
if (input > 0 || !output) {
  if (!(output && queuelength < 100)) {
    P1
  } else if (output && !(queuelength < 100)) {
    P2
  } else {
        P3
  }
} else {
  P4
}
```

Define the propositional variables:

- i: input > 0
- u: output
- q: queuelength < 100

The code fragment becomes the following. We'll call this code fragment #1.

```
if ( i || !u ) {
  if ( !(u && q) ) {
    P1
  } else if ( u && !q ) {
    P2
  } else { P3 }
} else { P4 }
```

Code fragment #2:

```
if (( i && u) && q) {
  P3
} else if (!i && u) {
  P4
} else {
  P1
}
```

Prove that these two pieces of code fragments are equivalent:

Prove that the condition leading to P_2 is logically equivalent to `F`.
The condition leading to P_2:

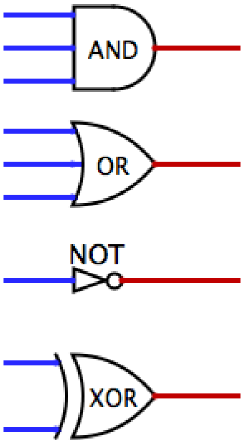
Prove that the condition leading to P_3 is true if and only if all three variables are true.
The condition leading to P_3:

Prove that the condition leading to P_4 is true if and only if i is false and u is true.
The condition leading to P_4:

The condition leading to P_1:

1.6 Circuit Design

Basic gates:



Problem: Your instructors, Alice, Carmen, and Collin, are choosing questions to be put on the midterm. For each problem, each instructor votes either yes or not. A question is chosen if it receives two or more yes votes. Design a circuit, which outputs yes whenever a question is chosen.

1. Draw the truth table based on the problem description.

x	y	z	output
T	T	T	
T	T	F	
T	F	T	
T	F	F	
F	T	T	
F	T	F	
F	F	T	
F	F	F	

2. Convert the truth table a propositional formula.

3. Then, convert the formula to a circuit.

Solution 1:

1. Convert the truth table a propositional formula.

2. Draw the circuit.

Solution 2:

1. Convert the truth table a propositional formula.

2. Draw the circuit.

Solution 3:

1. Convert the truth table a propositional formula.

2. Draw the circuit.

1.7 Semantic Entailment

Exercise 9. *Let* $\Sigma = \{(\mathrm{p} \to \mathrm{q}), (\mathrm{q} \to \mathrm{r})\}$. *Is* Σ *satisfiable? Why or why not?*

Exercise 10. *Let* $\Sigma = \emptyset$. *Is* Σ *satisfiable? Why or why not?*

Exercise 11. *Let* $\Sigma = \{\mathrm{p}, (\neg \mathrm{p})\}$. *Is* Σ *satisfiable? Why or why not?*

Exercise 12. *Prove that* $\{(\neg(\mathrm{p} \wedge \mathrm{q})), (\mathrm{p} \rightarrow \mathrm{q})\} \models (\neg \mathrm{p})$.

Exercise 13. *Prove that* $\{(\neg(\mathrm{p} \wedge \mathrm{q})), (\mathrm{p} \rightarrow \mathrm{q})\}$ 6$\models (\mathrm{p} \leftrightarrow \mathrm{q})$.

Exercise 14. *Prove that* $\emptyset \models ((\mathrm{p} \wedge \mathrm{q}) \rightarrow \mathrm{p}))$.

Exercise 15. *Prove that* $\{\mathrm{r}, (\mathrm{p} \to (\mathrm{r} \to \mathrm{q}))\} \models (\mathrm{p} \to (\mathrm{q} \land \mathrm{r}))$.

Exercise 16. *Prove that* $\{(\neg \mathrm{p}), (\mathrm{q} \to \mathrm{p})\}\ 6\models ((\neg \mathrm{p}) \land \mathrm{q})$.

Exercise 17. *Prove that* $\{\mathrm{p}, (\neg \mathrm{p})\} \models \mathrm{r}$.

1.8 Natural Deduction

1.8.1 Strategies for writing a natural deduction proof

General strategies:

- Write down all of the premises
- Leave plenty of space. Then write down the conclusion.
- Look at the conclusion carefully. What is the structure of the conclusion (what is the last connective applied in the formula? Can you apply an introduction rule to produce the conclusion?
- Look at each premise carefully. What is the structure of the premise (what is the last connective applied in the formula)? Can you apply an elimination rule to simplify it?
- Working backwards from the conclusion is often more effective than working forward from the premises. It keeps your eyes on the prize.
- If no rule is applicable, consider using a combination of ¬i and ¬¬e.

Working with subproofs

- To apply an introduction rule to produce the conclusion, **lay down the structure of the subproof before you proceed to fill in the subproof**. That is, draw the box for the subproof, write down the assumption on the first line, copy the conclusion to the last line of the subproof.
- Every subproof must be created to apply a particular rule. If you don't know what rule you are trying to apply, don't create a subproof.
- When filling in a subproof, you can use all the formulas that come before as long as the formula is not in a previous subproof that has already closed.
- Outside of a subproof, you have to use the subproof as a whole. You cannot use any individual formula in the subproof.

1.8.2 And elimination and introduction

Exercise 18. *Show that* $\{(\mathrm{p} \wedge \mathrm{q}), (\mathrm{r} \wedge \mathrm{s})\} \vdash (\mathrm{q} \wedge \mathrm{s})$.

Exercise 19. *Show that* $((\mathrm{p} \wedge \mathrm{q}) \wedge \mathrm{r}) \vdash (\mathrm{p} \wedge (\mathrm{q} \wedge \mathrm{r}))$.

1.8.3 Implication introduction and elimination

Exercise 20. *Show that* $\{(\mathrm{p} \to \mathrm{q}), (\mathrm{q} \to \mathrm{r})\} \vdash (\mathrm{p} \to \mathrm{r})$.

Exercise 21. *Show that* $\{(\mathrm{p} \to (\mathrm{q} \to \mathrm{r})), (\mathrm{p} \to \mathrm{q})\} \vdash (\mathrm{p} \to \mathrm{r})$.

Exercise 22. *Show that* $\{(\mathrm{p} \to (\mathrm{q} \to \mathrm{r}))\} \vdash ((\mathrm{p} \land \mathrm{q}) \to \mathrm{r})$.

Exercise 23. *Show that* $\{((\mathrm{p} \land \mathrm{q}) \to \mathrm{r})\} \vdash (\mathrm{p} \to (\mathrm{q} \to \mathrm{r}))$.

1.8.4 Or elimination and introduction

Exercise 24. *Show that* $\{(\mathrm{p} \wedge (\mathrm{q} \vee \mathrm{r}))\} \vdash ((\mathrm{p} \wedge \mathrm{q}) \vee (\mathrm{p} \wedge \mathrm{r}))$*.*

Exercise 25. *Show that* $\{((\mathrm{p} \wedge \mathrm{q}) \vee (\mathrm{p} \wedge \mathrm{r}))\} \vdash (\mathrm{p} \wedge (\mathrm{q} \vee \mathrm{r}))$*.*

Exercise 26. *Show that* $\{(\mathrm{p} \vee \mathrm{q})\} \vdash ((\mathrm{p} \rightarrow \mathrm{q}) \vee (\mathrm{q} \rightarrow \mathrm{p}))$.

Exercise 27. *Show that* $\{(\mathrm{p} \rightarrow \mathrm{q})\} \vdash ((\mathrm{r} \vee \mathrm{p}) \rightarrow (\mathrm{r} \vee \mathrm{q}))$.

1.8.5 Negation introduction and double negation elimination

Exercise 28. *Show that* $\{(\mathrm{p} \to (\neg \mathrm{p}))\} \vdash (\neg \mathrm{p})$.

Exercise 29. *Show that* $\{(\mathrm{p} \to (\mathrm{q} \to \mathrm{r})), \mathrm{p}, (\neg \mathrm{r})\} \vdash (\neg \mathrm{q})$.

Exercise 30. *Show that* $\{((\neg \mathrm{p}) \to (\neg \mathrm{q}))\} \vdash (\mathrm{q} \to \mathrm{p})$.

Exercise 31. *Show that* $\{((\mathrm{p} \wedge (\neg \mathrm{q})) \to \mathrm{r}), (\neg \mathrm{r}), \mathrm{p}\} \vdash \mathrm{q}$.

1.8.6 Negation elimination

Exercise 32. *Show that* $\{(\mathrm{p} \vee \mathrm{q}), (\neg \mathrm{p})\} \vdash \mathrm{q}$.

Exercise 33. *Show that* $\emptyset \vdash ((\neg \mathrm{p}) \to (\mathrm{p} \to (\mathrm{p} \to \mathrm{q})))$.

1.8.7 Putting it together!

Exercise 34. *(De Morgan's Law) Show that* $\{(\neg(\alpha \vee \beta))\} \vdash ((\neg\alpha) \wedge (\neg\beta))$.

Exercise 35. *(De Morgan's Law) Show that* $\{((\neg\alpha) \wedge (\neg\beta))\} \vdash (\neg(\alpha \vee \beta))$.

Exercise 36. *(De Morgan's Law) Show that* $\{((\neg\alpha) \vee (\neg\beta))\} \vdash (\neg(\alpha \wedge \beta))$.

Exercise 37. *(De Morgan's Law) Show that* $\{(\alpha \vee \beta)\} \vdash (\neg((\neg\alpha) \wedge (\neg\beta)))$.

Exercise 38. *(De Morgan's Law) Show that* $\{(\neg(\alpha \wedge \beta))\} \vdash ((\neg\alpha) \vee (\neg\beta))$.

Exercise 39. *Show that* $\{(\neg(p \to q))\} \vdash (q \to p)$.

1.8.8 Other problems

Exercise 40. *E4 Exercise 4: Prove that for any set of propositional formulas* Σ *and any propositional variables* p *and* q*, if* $\Sigma \vdash \mathrm{p}$*, then* $\Sigma \vdash ((\neg \mathrm{p}) \to \mathrm{q})$*.*

1.9 Soundness and Completeness of Natural Deduction

1.9.1 The soundness of inference rules

Exercise 41. *The following inference rule is called Disjunctive syllogism.*

$$\frac{(\neg\alpha) \quad (\alpha \vee \beta)}{\beta} \textit{ Disjunctive syllogism}$$

where α *and* β *are well-formed propositional formulas.*

Prove that this inference rule is sound. That is, prove that the following semantic entailment holds.

$$\{(\neg\alpha), (\alpha \vee \beta)\} \vDash \beta$$

You must use **the definition of semantic entailment** *to write your proof. Do not use any other technique such as truth table, valuation tree, logical identities, natural deduction, soundness, or completeness.*

Exercise 42. *Consider the following inference rule:*

$$\frac{(\alpha \to \beta)}{(\beta \to \alpha)} \textit{ Flip the implication}$$

where α *and* β *are well-formed propositional formulas.*

Prove that this inference rule is NOT sound. That is, prove the following statement:

$$\{(\alpha \to \beta)\} \not\models (\beta \to \alpha)$$

You must use **the definition of semantic entailment** *to write your proof. Do not use any other technique such as truth table, valuation tree, logical identities, natural deduction, soundness, or completeness.*

1.9.2 Soundness and Completeness of Natural Deduction

Exercise 43. *Prove or disprove this statement: If* $\{\mathrm{a}, \mathrm{b}\} \vdash \mathrm{c}$*, then* $\emptyset \vDash ((\mathrm{a} \wedge \mathrm{b}) \rightarrow \mathrm{c})$.

Exercise 44. *Prove or disprove this statement: If* $\{\alpha\} \models \beta$*, then* $\emptyset \vdash (\beta \to \alpha)$*.* α *and* β *are well-formed propositional formulas.*

2 Predicate Logic

2.1 Translations

Exercise 45. *Let the domain be the set of animals. Let* $\mathrm{B(x)}$ *mean that* x *is a bear. Let* $\mathrm{H(x)}$ *mean that* x *likes honey.*

Translate "every bear likes honey" into predicate logic.

Exercise 46. *Let the domain be the set of animals. Let* B(x) *mean that* x *is a bear. Let* H(x) *mean that* x *likes honey.*

Translate "some bear likes honey" into predicate logic.

Exercise 47. *Could you summarize the general patterns of translations based on the two exercises above? Which binary connectives usually go with the universal and the existential quantifiers?*

Alice: I put this exercise here so that I will have a place to put down a summary.

Exercise 48. *Translate the following sentences into predicate formulas.*

Let the domain contain the set of all students and courses. Define the following predicates:
C(x): x is a course.
S(x): x is a student.
T(x, y): student x has taken course y.

1. Every student has taken some course.

2. A student has taken a course.

3. No student has taken every course.

4. Some student has not taken any course.

5. Every student has taken every course.

Exercise 49. *Translating "at least", "at most", and "exactly". Translate the following sentences into predicate formulas.*

- There is at least one bear.
- There are at least two bears.
- There is at most one bear.
- There is exactly one bear.

2.2 Semantics of Predicate Formulas

Consider this language of predicate logic:

- Constant symbols: $\mathrm{a}, \mathrm{b}, \mathrm{c}$
- Variable symbols: $\mathrm{x}, \mathrm{y}, \mathrm{z}$
- Function symbols: $\mathrm{f}^{(1)}, \mathrm{g}^{(2)}$
- Predicate symbols: $\mathrm{P}^{(1)}, \mathrm{Q}^{(2)}$

2.2.1 Evaluating Formulas with No Variables

Exercise 50. *Give an interpretation* I *such that* $\mathrm{Q}(\mathrm{f}(\mathrm{c}), \mathrm{a})^{\mathrm{I}} = \mathtt{T}$ *where* $\mathrm{dom}(\mathrm{I}) = \{1, 2, 3\}$.

Exercise 51. *Give an interpretation* I *such that* $\mathrm{Q}(\mathrm{f}(\mathrm{c}), \mathrm{a})^{\mathrm{I}} = \mathtt{F}$.

2.2.2 Evaluating Formulas with Free Variables Only

Exercise 52. *Give an interpretation* I *and an environment* E *such that* $\mathrm{Q}(\mathrm{f}(\mathrm{x}), \mathrm{a})^{(\mathrm{I},\mathrm{E})} = \mathtt{T}$.

2.2.3 Evaluating Formulas with Free and Bound Variables

Exercise 53. *Give an interpretation* I *and an environment* E *such that* $\mathrm{I} \models_{\mathrm{E}} \alpha$ *where* α *is* $(\exists \mathrm{x}\ \mathrm{Q}(\mathrm{x}, \mathrm{y}))$*. Start with the domain* $\mathrm{dom}(\mathrm{I}) = \{1, 2, 3\}$.

Exercise 54. *Give an interpretation* I *and an environment* E *such that* $\mathrm{I} \models_{\mathrm{E}} \alpha$ *where* α *is* $(\forall \mathrm{x}\ \mathrm{Q}(\mathrm{x}, \mathrm{y}))$.

2.2.4 Evaluating Formulas with Bound Variables Only

Exercise 55. *Give an interpretation* I *and an environment* E *such that* $\mathrm{I} \models_{\mathrm{E}} (\exists \mathrm{x}(\forall \mathrm{y}\ \mathrm{Q}(\mathrm{x}, \mathrm{y})))$. *Start with the domain* $\mathrm{dom}(\mathrm{I}) = \{1, 2, 3\}$.

Exercise 56. *Give an interpretation* I *and an environment* E *such that* $\mathrm{I} \not\models_{\mathrm{E}} (\exists \mathrm{x}(\forall \mathrm{y}\ \mathrm{Q}(\mathrm{x}, \mathrm{y})))$. *Start with the domain* $\mathrm{dom}(\mathrm{I}) = \{1, 2, 3\}$.

2.3 Semantic Entailment

Exercise 57. *Show that* $\{(\forall \mathrm{x}\ \mathrm{P}(\mathrm{x}))\} \vDash (\exists \mathrm{x}\ \mathrm{P}(\mathrm{x}))$.

Exercise 58. *Show that* $\{(\exists \mathrm{x}\ \mathrm{P}(\mathrm{x}))\} \nvDash (\forall \mathrm{x}\ \mathrm{P}(\mathrm{x}))$.

Exercise 59. *Show that* $\{(\exists \mathrm{y}\ (\forall \mathrm{x}\ \mathrm{Q}(\mathrm{x},\mathrm{y})))\} \vDash (\forall \mathrm{x}\ (\exists \mathrm{y}\ \mathrm{Q}(\mathrm{x},\mathrm{y})))$.

Exercise 60. *Show that* $\{(\forall \mathrm{x}\ (\exists \mathrm{y}\ \mathrm{Q}(\mathrm{x}, \mathrm{y})))\} \not\models (\exists \mathrm{y}\ (\forall \mathrm{x}\ \mathrm{Q}(\mathrm{x}, \mathrm{y})))$.

Exercise 61. *Show that* $\{(\forall \mathrm{x}\ (\alpha \to \beta))\} \vDash ((\forall \mathrm{x}\ \alpha) \to (\forall \mathrm{x}\ \beta))$*, where* x *is a variable symbol and* α *and* β *are well-formed predicate formulas.*

Exercise 62. *Show that* $\{((\forall \mathrm{x}\ \alpha) \rightarrow (\forall \mathrm{x}\ \beta))\} \not\models (\forall \mathrm{x}\ (\alpha \rightarrow \beta))$*, where* x *is a variable symbol and* α *and* β *are well-formed predicate formulas.*

2.4 Natural Deduction

- ∀e (analogous to ∧e)
- ∀i (analogous to ∧i)
  - We know nothing about the fresh variable u except that u is a domain element. (If u is special, our conclusion may not be valid.)
  - The fresh variable u cannot escape the subproof box. For example, we cannot conclude α[u/x] outside of the box.
  - When you choose the fresh variable u, make sure that it has not appears anywhere outside of the subproof box in the proof.
- ∃e (analogous to ∨e)
  - Proof by cases.
  - The conclusion may have nothing to do with the starting formula.
- ∃i (analogous to ∨i)

2.4.1 Forall-elimination

Exercise 63. *Show that* $\{\mathrm{P(t)}, (\forall \mathrm{x}\ (\mathrm{P(x)} \to (\neg \mathrm{Q(x)})))\} \vdash (\neg \mathrm{Q(t)})$.

2.4.2 Exists-introduction

Exercise 64. *Show that* $\{(\neg \mathrm{P(y)})\} \vdash (\exists \mathrm{x}\ (\mathrm{P(x)} \to \mathrm{Q(y)}))$.

Exercise 65. *Show that* $\{(\forall \mathrm{x}\ \mathrm{P(x)})\} \vdash (\exists \mathrm{y}\ \mathrm{P(y)})$.

2.4.3 Forall-introduction

Exercise 66. *Show that* $\{(\forall \mathrm{x}\ \mathrm{P}(\mathrm{x}))\} \vdash (\forall \mathrm{y}\ \mathrm{P}(\mathrm{y}))$.

Exercise 67. *Show that* $\{(\forall \mathrm{x}\ (\mathrm{P}(\mathrm{x}) \to \mathrm{Q}(\mathrm{x}))), (\forall \mathrm{x}\ \mathrm{P}(\mathrm{x}))\} \vdash (\forall \mathrm{x}\ \mathrm{Q}(\mathrm{x}))$.

Exercise 68. *Show that* $\{(\forall \mathrm{x}\ (\mathrm{P}(\mathrm{x}) \to \mathrm{Q}(\mathrm{x})))\} \vdash ((\forall \mathrm{x}\ \mathrm{P}(\mathrm{x})) \to (\forall \mathrm{y}\ \mathrm{Q}(\mathrm{y})))$.

2.4.4 Exists-elimination

Exercise 69. *Show that* $\{(\exists \mathrm{x}\ \mathrm{P}(\mathrm{x}))\} \vdash (\exists \mathrm{y}\ \mathrm{P}(\mathrm{y}))$.

Exercise 70. *Show that* $\{(\forall \mathrm{x}\ (\mathrm{P}(\mathrm{x}) \to \mathrm{Q}(\mathrm{x}))), (\exists \mathrm{x}\ \mathrm{P}(\mathrm{x}))\} \vdash (\exists \mathrm{x}\ \mathrm{Q}(\mathrm{x}))$.

Exercise 71. *Show that* $\{(\forall \mathrm{x}\ (\mathrm{Q}(\mathrm{x}) \to \mathrm{R}(\mathrm{x}))), (\exists \mathrm{x}\ (\mathrm{P}(\mathrm{x}) \wedge \mathrm{Q}(\mathrm{x})))\} \vdash (\exists \mathrm{x}\ (\mathrm{P}(\mathrm{x}) \wedge \mathrm{R}(\mathrm{x})))$.

2.4.5 Putting them together

Exercise 72. *Show that* $\{(\exists \mathrm{x}\ \mathrm{P}(\mathrm{x})), (\forall \mathrm{x}\ (\forall \mathrm{y}\ (\mathrm{P}(\mathrm{x}) \to \mathrm{Q}(\mathrm{y}))))\} \vdash (\forall \mathrm{y}\ \mathrm{Q}(\mathrm{y}))$.

Exercise 73. *Show that* $\{(\exists \mathrm{y}\ (\forall \mathrm{x}\ \mathrm{P}(\mathrm{x}, \mathrm{y})))\} \vdash (\forall \mathrm{x}\ (\exists \mathrm{y}\ \mathrm{P}(\mathrm{x}, \mathrm{y})))$.

Exercise 74. *Show that* $\{(\neg(\exists \mathrm{x}\ \mathrm{P}(\mathrm{x})))\} \vdash (\forall \mathrm{x}\ (\neg \mathrm{P}(\mathrm{x})))$. *(De Morgan)*

Exercise 75. *Show that* $\{(\forall \mathrm{x}\ (\neg \mathrm{P}(\mathrm{x})))\} \vdash (\neg(\exists \mathrm{x}\ \mathrm{P}(\mathrm{x})))$. *(De Morgan)*

Exercise 76. *Show that* $\{(\exists \mathrm{x}\ (\neg \mathrm{P}(\mathrm{x})))\} \vdash (\neg(\forall \mathrm{x}\ \mathrm{P}(\mathrm{x})))$*. (De Morgan)*

Exercise 77. *Show that* $\{(\neg(\forall \mathrm{x}\ \mathrm{P}(\mathrm{x})))\} \vdash (\exists \mathrm{x}\ (\neg \mathrm{P}(\mathrm{x})))$*. (De Morgan)*

2.5 Soundness and Completeness of Natural Deduction

2.5.1 Proving that an inference rule is sound or not sound

Lemma 1. *Let* t *be a predicate term. Let* I *be an interpretation with domain* $\mathrm{dom(I)}$*. Let* E *be an environment. Then we have that* $\mathrm{t}^{(\mathrm{I},\mathrm{E})} \in \mathrm{dom(I)}$*.*

Lemma 2. *Let* α *be a well-formed predicate formula. Let* t *be a predicate term. Let* I *and* E *be an interpretation and environment. Let* x *be a variable. Then we have that* $\alpha[\mathrm{t}/\mathrm{x}]^{(\mathrm{I},\mathrm{E})} = \alpha^{(\mathrm{I},\mathrm{E}[\mathrm{x}\mapsto \mathrm{t}^{(\mathrm{I},\mathrm{E})}])}$*.*

Exercise 78. *Prove that the* $\forall e$ *inference rule is sound. That is, prove that the entailment holds:*

$$\{(\forall \mathrm{x}\ \alpha)\} \vDash \alpha[\mathrm{t}/\mathrm{x}] \tag{1}$$

where α *be a Predicate formula,* x *is a variable, and* t *is a Predicate term.*

Exercise 79. *Prove that the* $\exists i$ *inference rule is sound. That is, prove that the entailment holds:*

$$\{\alpha[\mathrm{t}/\mathrm{x}]\} \vDash (\exists \mathrm{x}\ \alpha) \tag{2}$$

where α *is a predicate formula,* t *is a predicate term, and* x *is a variable.*

Exercise 80. *Prove that the following inference rule is NOT sound.*

$$\frac{\mathrm{\alpha[t/x]}}{(\forall \mathrm{x}\ \mathrm{\alpha})}\ \forall \mathrm{i}* \tag{3}$$

where α *is a predicate formula,* t *is a predicate term, and* x *is a variable.*

Exercise 81. *Prove that the following inference rule is NOT sound.*

$$\frac{(\forall \mathrm{x}(\alpha \to \beta)) \quad \beta[\mathrm{t}/\mathrm{x}]}{\alpha[\mathrm{t}/\mathrm{x}]}\ \forall\mathrm{e}* \tag{4}$$

where α *and* β *are predicate formulas,* t *is a predicate term, and* x *is a variable.*