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A template for structural induction on well-formed formulas.

Problem: Prove that every well-formed formula φ has property P

Proof by structural induction:

Base case: φ is a propositional variable p.
We need to prove that p has property P.

Induction step:

Case 1: φ is a well-formed formula of the form $(\neg\alpha)$ for a well-formed formula α.

Induction hypothesis: Assume that α has property P.
We need to prove that $\varphi = (\neg\alpha)$ has property P.

Case 2: φ is a well-formed formula of the form $(\alpha \circledast \beta)$ for well-formed formulas α and β, and a binary connective $\circledast$ which is one of $\wedge, \vee, \rightarrow$, and $\leftrightarrow$.

Induction hypothesis: Assume that α and β have property P.
We need to prove that $\varphi = (\alpha \circledast \beta)$ has property P.

By the principle of structural induction, every well-formed formula φ has property P.

QED

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Comments:

① If we define $P(\varphi)$ to mean that φ has the property P, then we can replace every occurrence of "φ has property P" with "$P(\varphi)$ holds".

For example: "α has property P" becomes "$P(\alpha)$ holds".

② I used $\otimes$ to denote one of the four binary connectives. You could expand case 2 into 4 separate cases, one for each binary connective.

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Theorem: Every well-formed formula has an equal number of left and right brackets.

Definition of a well-formed formula:
A well-formed formula is one of
① a single propositional variable,
② $(\neg \alpha)$ where α is a well-formed formula,
③ one of $(\alpha \wedge \beta)$, $(\alpha \vee \beta)$, $(\alpha \rightarrow \beta)$, and $(\alpha \Leftrightarrow \beta)$ where α and β are well-formed formulas.

Let $P(\varphi)$ denote that a well-formed formula φ has an equal number of left and right brackets.

Theorem: $P(\varphi)$ holds for every well-formed formula φ.

Proof by structural induction:
Base case: φ is a propositional variable. Prove $P(\varphi)$ holds.

Inductive step: Consider an arbitrary well-formed formula φ. that is not a propositional variable.

Case 1: $\varphi = (\neg \alpha)$ where α is a well-formed formula.
Induction hypothesis: assume $P(\alpha)$ holds.
We need to prove $P(\varphi)$ holds.

Case 2: $\varphi = (\alpha \circledast \beta)$ for well-formed formulas α and β, and $\circledast$ is one of $\wedge$, $\vee$, $\rightarrow$, and $\Leftrightarrow$.
Induction hypothesis: assume $P(\alpha)$ and $P(\beta)$ hold.
We need to prove $P(\varphi)$ holds.

By the principle of structural induction, $P(\varphi)$ holds for every well-formed formula φ. QED

Theorem: Every proper prefix of a well-formed formula has more left brackets than right brackets.

Proof by structural induction:

Let $P(\varphi)$ be the property that every proper prefix of a well-formed formula φ has more left brackets than right brackets.

Base case: φ is a propositional variable. Prove that $P(\varphi)$ holds.

Inductive step: Consider an arbitrary well-formed formula φ that is not a propositional formula.

case 1: $\varphi = (\neg \alpha)$ for some well-formed formula α.
induction hypothesis: assume $P(\alpha)$ holds.
We need to prove that $P(\varphi)$ holds.

case 2: $\varphi = (\alpha \circledast \beta)$ for well-formed formulas α and β, and binary connective $\circledast \in \{\wedge, \vee, \rightarrow, \leftrightarrow\}$.
induction hypothesis: assume $P(\alpha)$ and $P(\beta)$ hold.
We need to prove that $P(\varphi)$ holds.

By the principle of structural induction, $P(\varphi)$ holds for any well-formed formula φ.

QED.

Theorem: There is a unique way to construct every well-formed formula.

Proof by structural induction:

Let φ be a well-formed formula.
We want to prove that there is a unique way to construct φ as a well-formed formula.

Base case: φ is a propositional variable. Clearly there is a unique way to construct φ.

Induction step:

case 1: φ is $(\neg\alpha)$ for a well-formed formula α.

Induction hypothesis: assume that there is a unique way to construct α as a well-formed formula.
We need to prove that there is a unique way to construct $(\neg\alpha)$.

We already know one way to construct $(\neg\alpha)$: construct α first, then apply negation at the last step.
We want to show that there is no other way to construct $(\neg\alpha)$. Let's prove this by contradiction.

Suppose that there is another way to construct $(\neg\alpha)$.
$(\neg\alpha)$ cannot be a propositional variable because it has at least 3 symbols.
$(\neg\alpha)$ has to be constructed by applying a binary connective $\circledast$ at the last step.

Induction step (continued).

Thus, $(\neg\alpha) = (p \circledast q)$. So $p = \neg r$ where r is a proper prefix of α. By Lemma 2, r does not have balanced brackets. By Lemma 1, p is not a formula. Therefore, this construction is not valid.

case 2: φ is $(\alpha \circledast \beta)$ for well-formed formulas α and β, and a binary connective $\circledast$ (one of $\wedge, \vee, \rightarrow$, and $\Leftrightarrow$).

Induction hypothesis: assume that there is a unique way to construct α (and β) as a well-formed formula.

We need to prove that there is a unique way to construct $(\alpha \circledast \beta)$.

We already know one way to construct $(\alpha \circledast \beta)$:

construct α and β separately, and apply $\circledast$ at the last step.

We want to show that there is no other way to construct $(\alpha \circledast \beta)$. Let's prove this by contradiction.

Suppose that there is another way to construct $(\alpha \circledast \beta)$.

① as a propositional variable? impossible

② as $(\neg k)$

only works when α starts with $\neg$.

show that k is not well-formed.

③ as $(\alpha' \circledast' \beta')$

show that α' is not well-formed.