CS 245 – Examples of structural induction proofs September 14, 2017
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Lemma 2: Every well-formed formula has an equal number of opening and closing brackets. (This is proved in a separate handout. For this proof, we assume that this lemma is true.)

Lemma 3: Every proper prefix of a well-formed formula φ has more opening than closing brackets.

Define $P(\varphi)$ to be that every proper prefix of φ has more opening than closing brackets.

Proof by structural induction:

Base case: φ is a propositional variable. We need to prove that $P(\varphi)$ holds.
A propositional variable has no proper prefix. Thus, the theorem is true and $P(\varphi)$ holds.

Induction step:

Let $op(x)$ and $cl(x)$ denote the number of opening and closing brackets in x respectively.

Case 1: φ is a well-formed formula of the form $(\neg x)$ where x is a well-formed formula.

Induction hypothesis: Assume that $P(x)$ holds. Let m denote any proper prefix of x. The induction hypothesis becomes that m has more opening than closing brackets.
We need to prove that $P((\neg x))$ holds.

There are four possible proper prefixes of $(\neg x)$: (, $(\neg$, $(\neg m$, and $(\neg x$. We'll prove the four cases separately below.

Case a: $op(\,(\,) = 1$ by inspection. $cl(\,(\,) = 0$ by inspection. $op(\,(\,) > cl(\,(\,)$

Case b: $op(\,(\neg\,) = 1$ by inspection. $cl(\,(\neg\,) = 0$ by inspection. $op(\,(\neg\,) > cl(\,(\neg\,)$

Case c: $op(\,(\neg m\,)$

$= 1 + op(m)$	by inspection of $(\neg m$
$> 1 + cl(m)$	by the induction hypothesis
$> cl(m)$	algebra
$= cl(\,(\neg m\,)$	by inspection of $(\neg m$

Case d:

$op(\,(\neg x\,)$

$= 1 + op(x)$	by inspection $(\neg x$
$= 1 + cl(x)$	by Lemma 2 and x is a well-formed formula
$> cl(x)$	algebra
$= cl(\,(\neg x\,)$	by inspection $(\neg x$

Thus, $P((\neg x))$ holds.

(Proof continued on the next page)

Case 2: φ is a well-formed formula of the form $(x * y)$ where x and y are well-formed formulas and * is one of the four binary connectives ($\wedge$, $\vee$, $\rightarrow$, and $\leftrightarrow$).

Induction hypothesis: Assume that $P(x)$ and $P(y)$ hold. Let m and n denote any proper prefix of x and y respectively. The induction hypothesis becomes that each of m and n has more opening than closing brackets.
We need to prove that $P((x * y))$ holds.

There are six possible proper prefixes of $(x * y)$: (, $(m$, $(x$, $(x *$, $(x * n$, and $(x * y$. We'll prove the six cases separately below.

Case a: $op(( ) = 1$ by inspection. $cl(( ) = 0$ by inspection. $op(( ) > cl(( )$

Case b: $op((m)$

$= 1 + op(m)$	by inspection of $(m$
$> 1 + cl(m)$	by the induction hypothesis
$> cl(m)$	algebra
$= cl((m)$	by inspection of $(m$

Case c: $op((x)$

$= 1 + op(x)$	by inspection of $(x$
$= 1 + cl(x)$	by Lemma 2 and x is a well-formed formula
$> cl(x)$	algebra
$= cl((x)$	by inspection of $(x$

Case d: $op((x *)$

$= 1 + op(x)$	by inspection of $(x *$
$= 1 + cl(x)$	by Lemma 2 and x is a well-formed formula
$> cl(x)$	algebra
$= cl((x *)$	by inspection of $(x *$

Case e: $op((x * n)$

$= 1 + op(x) + op(n)$	by inspection $(x * n$
$= 1 + cl(x) + op(n)$	by Lemma 2 and x is a well-formed formula
$> 1 + cl(x) + cl(n)$	by the induction hypothesis
$> cl(x) + cl(n)$	algebra
$= cl((x * n)$	by inspection $(x * n$

Case f: $op((x * y)$

$= 1 + op(x) + op(y)$	by inspection $(x * y$
$= 1 + cl(x) + cl(y)$	by Lemma 2 and x and y are well-formed formulas
$> cl(x) + cl(y)$	algebra
$= cl((x * y)$	by inspection $(x * y$

By the principle of structural induction, $P(\varphi)$ holds for every well-formed formula φ. QED