Adequate Set of Connectives

Sept 21.

Theorem: $\{\wedge, \vee, \neg\}$ is an adequate set of connectives.

Proof by example: (This is used to illustrate the idea
DON'T DO THIS ANYWHERE ELSE! ☺)

$$(p \leftrightarrow q) \wedge (r \rightarrow q)$$

p	q	r	$p \leftrightarrow q$	$r \rightarrow q$	$(p \leftrightarrow q) \wedge (r \rightarrow q)$
0	0	0	1	1	1
0	0	1	1	0	0
0	1	0	0	1	0
0	1	1	0	1	0
1	0	0	0	1	0
1	0	1	0	0	0
1	1	0	1	1	1
1	1	1	1	1	1

p	q	r	$(p \leftrightarrow q) \wedge (r \rightarrow q)$	
0	0	0	1	$(\neg p) \wedge (\neg q) \wedge (\neg r)$
1	1	0	1	$p \wedge q \wedge (\neg r)$
1	1	1	1	$p \wedge q \wedge r$

$$((\neg p) \wedge (\neg q) \wedge (\neg r)) \vee (p \wedge q \wedge (\neg r)) \vee (p \wedge q \wedge r).$$

Give me any propositional logic formula, I can write it using $\wedge$, $\vee$, and $\neg$ by following this procedure

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Theorem: the set $\{\wedge, \vee, \neg\}$ is an adequate set of connectives.

Proof:

Consider an arbitrary propositional formula.
Construct the truth table for the formula.
Identify the rows for which the formula is true.
For each such row, construct a conjunction.
- If the variable is true, write it as is
- If the variable is false, put its negation into the conjunction

Then connect all the conjunctions using disjunctions.

QED.

(See a complete proof of one of following theorem on the next page.)

Theorem: $\{\wedge, \neg\}$ is adequate. (These are short & incomplete proofs.)

Proof:
$$(x \vee y) \equiv (\neg(\neg(x \vee y))) \quad \text{Double negation}$$
$$\equiv (\neg((\neg x) \wedge (\neg y))) \quad \text{De Morgan.}$$

Theorem: $\{\rightarrow, \neg\}$ is adequate.

Proof:
$$(x \vee y) \equiv ((\neg(\neg x)) \vee y) \quad \text{Double negation}$$
$$\equiv ((\neg x) \rightarrow y) \quad \text{Implication.}$$

$$(x \wedge y) \equiv (\neg(\neg(x \wedge y))) \quad \text{Double negation}$$
$$\equiv (\neg((\neg x) \vee (\neg y))) \quad \text{De Morgan.}$$
$$\equiv (\neg(x \rightarrow (\neg y))) \quad \text{Implication.}$$

Theorem: $\{\vee, \neg\}$ is adequate.

Proof:
$$(x \wedge y) \equiv (\neg(\neg(x \wedge y))) \quad \text{Double negation.}$$
$$\equiv (\neg((\neg x) \vee (\neg y))) \quad \text{De Morgan.}$$

Adequate set of connectives

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Theorem 1: $\{\wedge, \vee, \neg\}$ is an adequate set of connectives.

Theorem 2: $\{\wedge, \neg\}$ is adequate.

Proof: By theorem 1, $\{\wedge, \vee, \neg\}$ is an adequate set. So it suffices to show that $\vee$ is definable in terms of $\wedge$ and $\neg$.

Consider any well-formed formula $(x \vee y)$ where x and y are well-formed. We will show that $(x \vee y)$ is logically equivalent to $(\neg((\neg x) \wedge (\neg y)))$ using only $\wedge$ and $\neg$ as connectives.

$$
\begin{aligned}
& (x \vee y) & \\
\equiv\ & (\neg(\neg(x \vee y))) & \text{Double negation} \\
\equiv\ & (\neg((\neg x) \wedge (\neg y))) & \text{De Morgan's.}
\end{aligned}
$$

Thus, $\{\wedge, \neg\}$ is adequate.

QED