Semantic entailment.

Show that \( \vdash (p \to q), (q \to r) \vdash (p \to r) \)

**Proof 1:**

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The * marks all the rows in which \( p \to q \) and \( q \to r \) are both true. \( (p \to r) \) is true in all of the * rows. So the entailment holds. **QED.**

**Proof 2:** We prove this by contradiction.

Assume that the entailment does not hold.

There is a truth valuation \( t \) such that

\[
(p \to q)^t = T, \quad (q \to r)^t = T \quad \text{and} \quad (p \to r)^t = F.
\]

If \( (p \to r)^t = F \), then \( t \) has to be that \( p^t = T \) and \( r^t = F \).

If \( (p \to q)^t = T \) and \( p^t = T \), then \( q^t = T \). This is a contradiction.

If \( (q \to r)^t = T \) and \( r^t = F \), then \( q^t = F \) and \( r^t = F \).

Our assumption is false and the entailment holds. **QED.**
Semantic entailment

Show that
\[ \not \vDash (p \Rightarrow (-q)) \lor r, (q \land (-r)), (p \leftrightarrow r) \not\vDash (p \land (q \Rightarrow r)) \]

Proof: Consider a truth valuation \( t \) such that
\[ p^t = F, \quad q^t = T, \quad \text{and} \quad r^t = F. \]

\[(p \Rightarrow (-q)) \lor r)^t = ((T \Rightarrow F) \lor F) = T \]
\[(q \land (-r))^t = (T \land T) = T \]
\[(p \leftrightarrow r)^t = (F \leftrightarrow F) = T \]

\[(p \land (q \Rightarrow r))^t = (F \land (T \Rightarrow F)) = F \]

The premises are true but the conclusion is false, so the entailment does not hold. \( \Box \)