Derived rules for natural deduction.

1. Modus Tollens (MT) $\{(a \rightarrow b), (\neg b)\} \vdash (\neg a)$.

Proof:

1.	$(a \rightarrow b)$	premise
2.	$(\neg b)$	premise
3.	a	assumption
4.	b	$\rightarrow e$: 1,3
5.	$\bot$	$\bot i$: 2,4.
6.	$(\neg a)$	$\neg i$: 3–5.

QED.

2. Double negation introduction ($\neg\neg i$). $\{a\} \vdash (\neg(\neg a))$

Proof:

1.	a	premise
2.	$(\neg a)$	assumption
3.	$\bot$	$\neg e$: 1,2
4.	$(\neg(\neg a))$	$\neg i$: 2,3.

3. Proof by contradiction (PBC)

$$\frac{\boxed{\begin{array}{l}\neg a\\ \vdots\\ \bot\end{array}}}{a}\ \text{PBC}$$

Proof:

1.	$(\neg a) \rightarrow \bot$	given
2	$(\neg a)$	assumption
3	$\bot$	$\rightarrow e$: 1,2.
4	$(\neg(\neg a))$	$\neg i$: 2,3
5.	a	$\neg\neg e$: 4.

Derived rules for natural deduction

4. Law of excluded middle (LEM) $\emptyset \vdash (a \vee (\neg a))$

Proof:

1.	$(\neg(a \vee (\neg a)))$	assumption
2	a	assumption
3	$(a \vee (\neg a))$	$\vee i$: 2
4	$\perp$	$\neg e$: 1,3
5	$(\neg a)$	$\neg i$: 2-4
6	$(a \vee (\neg a))$	$\vee i$: 5
7	$\perp$	$\neg e$: 1,6
8	$(\neg(\neg(a \vee (\neg a))))$	$\neg i$: 1-7
9	$(a \vee (\neg a))$	$\neg\neg e$: 8.

QED

Natural deduction examples

① $\{p, (q \to r)\} \vdash ((p \land q) \to r)$

Proof:

1.	p	premise
2.	$q \to r$	premise
3.	$p \land q$	assumption
4	q	$\land e : 3$
5	r	$\to e : 2,4$
6	$((p \land q) \to r)$	$\to i : 3\text{–}5$.

QED

② De Morgan's law (direction ①)

$\{(\neg(p \lor q))\} \vdash ((\neg p) \land (\neg q))$.

Proof:

1.	$\neg(p \lor q)$	premise.
2	p	assumption
3	$(p \lor q)$	$\lor i : 2$
4	$\bot$	$\bot i : 1,3$.
5	$(\neg p)$	$\neg i : 2\text{–}4$
6	q	assumption
7	$(p \lor q)$	$\lor i : 6$
8	$\bot$	$\bot i : 1,7$
9	$(\neg q)$	$\neg i : 6\text{–}8$
10	$(\neg p) \land (\neg q)$	$\land i : 5,9$.

③ De Morgan's Law (direction ②)

$\{((\neg p) \wedge (\neg q))\} \vdash (\neg(p \vee q))$

Proof:

1.	$((\neg p) \wedge (\neg q))$	premise
2.	$(\neg p)$	$\wedge e$: 1
3.	$(\neg q)$	$\wedge e$: 1
4	$(p \vee q)$	assumption
5	p	assumption
6	$\perp$	$\perp i$: 2,5
7	q	assumption
8	$\perp$	$\perp i$: 3,7
9	$\perp$	$\vee e$: 4, 5-6, 7-8
10	$(\neg(p \vee q))$	$\neg i$: 4-9.

QED

④ $\{(p \to q)\} \vdash ((\neg p) \vee q)$

Proof:

1	$(p \to q)$	premise
2	$(\neg((\neg p) \vee q))$	assumption
3	$(\neg p)$	assumption
4	$((\neg p) \vee q)$	$\vee i$: 3
5	$\perp$	$\perp i$: 2,4
6	$(\neg(\neg p))$	$\neg i$: 3-5
7	p	$\neg\neg e$: 6
8	q	$\to e$: 1,7
9	$((\neg p) \vee q)$	$\vee i$: 8
10	$\perp$	$\perp i$: 2,9
11	$(\neg(\neg((\neg p) \vee q)))$	$\neg i$: 2-10
12	$((\neg p) \vee q)$	$\neg\neg e$: 11

Natural deduction examples

⑤ $\{(\alpha \vee \beta), (\neg\alpha)\} \vdash \beta$

Proof:

1.	$(\alpha \vee \beta)$	premise
2	$(\neg\alpha)$	premise
3	α	assumption
4	$\perp$	$\perp i$: 2,3
5	β	$\perp e$: 4
6	β	assumption
7	β	$\vee e$: 1, 3–5, 6

QED

⑥ $\{\} \vdash ((p \wedge q) \to p)$

Proof:

1.	$(p \wedge q)$	assumption
2	p	$\wedge e$: 1
3	$((p \wedge q) \to p)$	$\to i$: 1–2

QED

⑦ $\{\} \vdash (\alpha \vee (\neg\alpha))$ (This is a useful derived rule called the law of excluded middle.)

Proof:

1.	$(\neg(\alpha \vee (\neg\alpha)))$	assumption
2	α	assumption
3	$(\alpha \vee (\neg\alpha))$	$\vee i$: 2
4	$\perp$	$\perp i$: 1,3
5	$(\neg\alpha)$	$\neg i$: 2–4
6	$(\alpha \vee (\neg\alpha))$	$\vee i$: 5
7	$\perp$	$\perp i$: 1,6
8	$(\neg(\neg(\alpha \vee (\neg\alpha))))$	$\neg i$: 1–7
9	$(\alpha \vee (\neg\alpha))$	$\neg\neg e$: 8

QED

⑧ $\{(a \to b), (\neg b)\} \vdash (\neg a)$.

(This is a useful derived rule called "modus tollens".)

Proof:

1.	$(a \to b)$	premise
2.	$(\neg b)$	premise
3	a	assumption
4	b	$\to e$: 1, 3
5	$\perp$	$\perp i$: 2,4
6.	$(\neg a)$	$\neg i$: 3–5

QED