Assignment 4 Q1(b) $\{(q \vee r)\} \vdash ((q \wedge s) \vee (s \rightarrow r))$

① Start by writing down the premises and the conclusion.

Proof: 1. $(q \vee r)$ premise

$\vdots$

$((q \wedge s) \vee (s \rightarrow r))$

② Look at the premise. Can we apply any rule to it?
(most likely an elimination rule)

We could apply $\vee e$ to $(q \vee r)$.

Look at the conclusion. How can we reach it using any rule?
(most likely an introduction rule).

We could reach the conclusion by $\vee i$. This means we only need to derive one of $(q \wedge s)$ and $(s \rightarrow r)$ to reach the conclusion. But it's unclear which one we should aim to derive now.

Let's try applying $\vee e$ to $(q \vee r)$.

Proof: 1. $(q \vee r)$ premise

q	assumption
$\vdots$	
$((q \wedge s) \vee (s \rightarrow r))$	

r	assumption
$\vdots$	
$((q \wedge s) \vee (s \rightarrow r))$	

$((q \wedge s) \vee (s \rightarrow r))$ $\vee e$:

- Note that I set up the structure of $\vee e$ before filling in the details of the two subproofs
- The structure includes the assumption and the last line of each subproof and the line using $\vee e$.

③ Consider the second subproof from r to $((q \wedge s) \vee (s \rightarrow r))$.

To reach $((q \wedge s) \vee (s \rightarrow r))$, we need to prove one of $(q \wedge s)$ and $(s \rightarrow r)$. Knowing that r is true, it makes sense that $(s \rightarrow r)$ is true regardless of the truth value of s.

The subproof becomes:

r	assumption
$\vdots$	
$(s \rightarrow r)$	
$((q \wedge s) \vee (s \rightarrow r))$	$\vee i$

How do we prove $(s \rightarrow r)$ from r?

We could use $\rightarrow i$ to prove $(s \rightarrow r)$. The subproof becomes:

r	assumption
s	assumption
$\vdots$	
r	
$(s \rightarrow r)$	$\rightarrow i$:
$((q \wedge s) \vee (s \rightarrow r))$	$\vee i$:

We can derive r by reflexivity. Hooray!

r	assumption
s	assumption
r	reflexive:
$(s \rightarrow r)$	$\rightarrow i$
$((q \wedge s) \vee (s \rightarrow r))$	$\vee i$:

④ Let's try to complete the first subproof from q to $((q \land s) \lor (s \to r))$.

We need to prove one of $(q \land s)$ and $(s \to r)$.
- If q is true, we need to know s is true to prove $(q \land s)$.
- It's unclear how to prove $(s \to r)$ from q.

So it seems we cannot directly prove $((q \land s) \lor (s \to r))$ from q....

④a We could try using $\neg i$ and $\neg\neg e$. Let's give it a try.

q	assumption
$\quad (\neg((q \land s) \lor (s \to r)))$	assumption
$\quad \vdots$	
$\quad \bot$	
$(\neg(\neg((q \land s) \lor (s \to r))))$	$\neg i$:
$((q \land s) \lor (s \to r))$	$\neg\neg e$:

④b Note that both of $(q \land s)$ and $(s \to r)$ have an s in it.
Intuitively, if s is true, then $(q \land s)$ is true.
If s is false, then $(s \to r)$ is true.
Either way, we could conclude $((q \land s) \lor (s \to r))$.

This means that we could try $(s \lor (\neg s))$ by L.E.M. and use $\lor e$ on $(s \lor (\neg s))$.

q	assumption
$(s \lor (\neg s))$	LEM
$\quad s$	assumption
$\quad \vdots$	
$\quad ((q \land s) \lor (s \to r))$	
$\quad (\neg s)$	assumption
$\quad \vdots$	
$\quad ((q \land s) \lor (s \to r))$	
$((q \land s) \lor (s \to r))$	$\lor e$

(4a) continued. How do we reach a contradiction in the inner subproof? We could try to derive $((q \vee s) \vee (s \to r))$ in it.

Note that this is not circular reasoning. In the outer subproof, we could only use q to prove $((q \wedge s) \vee (s \to r))$. In the inner subproof, we could use q and $(\neg((q \wedge s) \vee (s \to r)))$ to derive $((q \wedge s) \vee (s \to r))$.

q	assumption
$(\neg((q \wedge s) \vee (s \to r)))$	assumption
$\vdots$	
$((q \wedge s) \vee (s \to r))$	
$\perp$	$\perp i$
$(\neg(\neg((q \wedge s) \vee (s \to r))))$	$\neg i$:
$((q \wedge s) \vee (s \to r))$	$\neg\neg e$:

(5a) To derive $((q \wedge s) \vee (s \to r))$, we need to derive one of $(q \wedge s)$ and $(s \to r)$.

We know q is true, but we don't know s is true. So we cannot prove $(q \wedge s)$. Let's try proving $(s \to r)$ using $\to i$.

q	assumption
$(\neg((q \wedge s) \vee (s \to r)))$	assumption
s	assumption
$\vdots$	
r	
$(s \to r)$	$\to i$:
$((q \wedge s) \vee (s \to r))$	$\vee i$:
$\perp$	$\perp i$:
$(\neg(\neg((q \wedge s) \vee (s \to r))))$	$\neg i$:
$((q \wedge s) \vee (s \to r))$	$\neg\neg e$:

(6a) How do we complete the innermost subproof?
Inside this subproof, we know q and s are both sure, so $(q \land s)$ is true by $\land i$. Then $((q \land s) \lor (s \to r))$ is true by $\lor i$. This is a contradiction to $(\neg((q \land s) \lor (s \to r)))$.
This contradiction allows us to derive anything such as r by $\perp e$. The complete proof is below.

Proof:

1	$(q \lor r)$	
2	q	assumption
3	$(\neg((q \land s) \lor (s \to r)))$	assumption
4	s	assumption
5	$(q \land s)$	$\land i$: 2,4
6	$((q \land s) \lor (s \to r))$	$\lor i$: 5
7	$\perp$	$\perp i$: 3,6
8	r	$\perp e$: 7
9	$(s \to r)$	$\to i$: 4-8
10	$((q \land s) \lor (s \to r))$	$\lor i$: 9
11	$\perp$	$\perp i$: 3, 10
12	$(\neg(\neg((q \land s) \lor (s \to r))))$	$\neg i$: 3-11
13	$((q \land s) \lor (s \to r))$	$\neg\neg e$: 12
14	r	assumption
15	s	assumption
16	r	reflexive: 14
17	$(s \to r)$	$\to i$: 15-16
18	$((q \land s) \lor (s \to r))$	$\lor i$: 17
19	$((q \land s) \lor (s \to r))$	$\lor e$: 1, 2-13, 14-18

QED

(4b) continued. ~ How do we complete the first inner subproof from s to $((q \wedge s) \vee (s \to r))$?

In this subproof, we know q and s are true. By $\wedge i$, $(q \wedge s)$ is true, so $((q \wedge s) \vee (s \to r))$ is true.

~ How do we complete the second inner subproof from $(\neg s)$ to $((q \wedge s) \vee (s \to r))$?

We know s is false, so $s \to r$ is vacuously true. We can prove $((q \wedge s) \vee (s \to r))$ by $\vee i$.

First inner subproof:

q	assumption
$(s \vee (\neg s))$	LEM
s	assumption
$(q \wedge s)$	$\wedge i$
$((q \wedge s) \vee (s \to r))$	$\vee i$

Second inner subproof:

$(\neg s)$	assumption
s	assumption
$\vdots$	
r	
$(s \to r)$	$\to i$:
$((q \wedge s) \vee (s \to r))$	$\vee i$:

To complete the second inner subproof, notice that $(\neg s)$ and s lead to a contradiction $\perp$, which allows us to derive anything such as r. by $\perp e$.

$(\neg s)$	assumption
s	assumption
$\perp$	$\perp i$:
r	$\perp e$:
$(s \to r)$	$\to i$:
$((q \wedge s) \vee (s \to r))$	$\vee i$:

(5b) Here is the complete proof using LEM.

Proof:

1.	$(q \lor r)$	
2	q	assumption
3	$(s \lor (\neg s))$	LEM
4	s	assumption
5	$(q \land s)$	$\land i$: 2,4
6	$((q \land s) \lor (s \to r))$	$\lor i$: 5
7	$(\neg s)$	assumption
8	s	assumption
9	$\perp$	$\perp i$: 7,8
10	r	$\perp e$: 9
11	$(s \to r)$	$\to i$: 8–10
12	$((q \land s) \lor (s \to r))$	$\lor i$: 11
13	$((q \land s) \lor (s \to r))$	$\lor e$: 3, 4-6, 7–12
14	r	assumption
15	s	assumption
16	r	reflexive : 15
17	$(s \to r)$	$\to i$: 15–16
18	$((q \land s) \lor (s \to r))$	$\lor i$: 17
19	$((q \land s) \lor (s \to r))$	$\lor e$: 1, 2–13, 14–18