Semantics of Predicate Logic

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The question: Given a well-formed predicate logic formula, is it T or F in some context?

In propositional logic, a truth valuation is enough to assign a meaning to a formula.
In predicate logic, we need a lot more.

Properties of formulas

① A formula α is valid if $I \vDash_E \alpha$ for every interpretation I and environment E.
- $I \vDash_E \alpha$ means "I and E make α true or satisfy α."
- I and E together make up the context.
- "valid" is analogous to "tautology" in prop logic.

② A formula α is satisfiable if $I \vDash_E \alpha$ for some interpretation I and environment E.

③ A formula α is unsatisfiable if $I \nvDash_E \alpha$ for every interpretation I and environment E.
- "unsatisfiable" is analogous to "contradiction".

Most predicate formulas are satisfiable but not valid because we have a great deal of freedom to choose I and E.

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Let's define a set of symbols first.

Constant symbols : a, b, c
Variable symbols : x, y, z.
Function symbols : $f^{(1)}, h^{(2)}$
Predicate symbols : $P^{(1)}, Q^{(2)}$

These symbols do not have intrinsic meanings. They only get meanings through interpretations and environments.

Recall that there are two kinds of expressions : terms and formulas.
Let's divide them into 3 categories.

Category 1 : terms and formulas without variables.
Examples : terms : $f(h(f(a), f(c)))$, $f(h(b, f(a)))$
formulas : $Q(f(c), a)$, $P(h(f(a), f(c)))$

We only need an interpretation to give these expressions meaning

Category 2 : terms and formulas with free variables only.
Examples : terms : $h(f(a), z)$ $f(h(y, c))$
formulas : $P(h(f(a), z))$ $Q(y, h(a, b))$.

We need an interpretation and an environment to give these expression meaning.

Category 3 : formulas with free and bound variables
Examples : $(\exists x (\forall y\ R(x, y, h(z, c))))$

Evaluating a Predicate Logic Formula.

An interpretation I consists of a domain and the meanings for all of the constant, function and predicate symbols.

Example: Interpretation I_1:

Domain $D = \{1, 2, 3\}$

Constants: $a^{I_1} = 1$, $b^{I_1} = 2$, $c^{I_1} = 3$

Functions: f^{I_1}: $f(1) = 2$, $f(2) = 3$, $f(3) = 1$.

h^{I_1}: $h(x, y) = \min(x, y)$, $\forall x, y \in D$.

Predicates: $P^{I_1} = \{1, 3\}$

$Q^{I_1} = \{\langle 1, 2\rangle, \langle 3, 3\rangle, \langle 3, 1\rangle\}$.

This notation means $Q(1,2)^{I_1} = T$, $Q(3,3)^{I_1} = T$, $Q(3,1)^{I_1} = T$ and for any other pair (a, b) in D^2, $Q(a,b)^{I_1} = F$.

An interpretation gives a meaning to every symbol even if our terms or formulas do not use the symbol.

Let's interpret the expressions in category 1:

$$f(h(f(a), f(c)))^{I_1} = 2$$

(Rough work $f(h(f(1), f(3)))^{I_1} = f(h(2, 1))^{I_1} = f(1)^{I_1} = 2$.)

$$f(h(b, f(a)))^{I_1} = 3$$

(Rough work $f(h(2, f(1)))^{I_1} = f(h(2,2))^{I_1} = f(2)^{I_1} = 3$)

$$Q(f(c), a)^{I_1} = F$$

(Rough work $Q(f(3), 1)^{I_1} = Q(1,1)^{I_1} = F$)

$$P(h(f(a), f(c)))^{I_1} = T$$

(Rough work $P(h(f(a), f(c)))^{I_1} = P(h(f(1), f(3)))^{I_1} = P(h(2,1))^{I_1} = P(1)^{I_1} = T$

Evaluating a Predicate Logic Formula

Questions:

We saw that $Q(f(c), a)^{I_1} = F$, so $I_1 \not\models Q(f(c), a)$.

Is there an interpretation I_2 such that $I_2 \models Q(f(c), a)$?

Let's start with $I_2 = I_1$ and make a small change to I_2 to answer this

$$Q(f(c), a)^{I_2} = Q(f(3), 1)^{I_2} = Q(1, 1)^{I_2}.$$

To make Q true, we only need to make sure that $\langle 1, 1 \rangle \in Q^{I_2}$. So let $Q^{I_2} = \{\langle 1, 1 \rangle\}$.

$$Q(f(c), a)^{I_2} = Q(f(3), 1)^{I_2} = Q(1, 1)^{I_2} = T. \text{ so } I_2 \models Q(f(c), a).$$

We saw that $P(h(f(a), f(c)))^{I_1} = T$.

Is there an interpretation I_3 such that $I_3 \not\models P(h(f(a), f(c)))$?

Let's start with $I_3 = I_1$.

Note that

$$P(h(f(a), f(c)))^{I_3} = P(h(f(1), f(3)))^{I_3} = P(h(2, 1))^{I_3}$$
$$= P(1)^{I_3}$$

To make this false, we only need to make sure that $1 \notin P^{I_3}$. Let's define $P^{I_3} = \phi$ (the empty set, this predicate P is always false.)

So $P(h(f(a), f(c)))^{I_3} = F$. $I_3 \not\models P(h(f(a), f(c)))$.

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A note about functions in an interpretation.

A function symbol $f^{(k)}$ must be interpreted as a function f^I that is total on D.

$$f^{(k)} : \underbrace{D \times \cdots \times D}_{k \text{ copies}} \to D$$

① Any k-tuple in D^k can be an input to $f^{(k)}$.

② The output of f^I must be in D.

Examples of non-total functions:

① $f(x, y) = x - y$ Domain $D = \mathbb{N}$ natural numbers.

$f(1, 2) = 1 - 2 = -1 \notin \mathbb{N}$.

② $f(x) = \sqrt{x}$ Domain $D = \mathbb{Z}$ integers.

(1) $-1 \in D$ cannot be an input to f.

(2) $f(2) = \sqrt{2} \notin D$

Examples of total functions

① $D = \{1, 2, 3\}$

$f(1) = 1,\ f(2) = 1,\ f(3) = 1$

② $D = \mathbb{N}$ natural numbers

$f(x) = x + 1 \quad \forall x \in \mathbb{N}$.

To evaluate terms and formulas with variables, we need an interpretation and an environment.

An environment maps every variable symbol to an element of the domain.
- An environment is only used to interpret free variables
- Bound variables get their meanings through the corresponding quantifiers

Example: Environment E_1:
$E_1(x) = 3$, $E_1(y) = 3$, $E_1(z) = 1$

Consider expressions in category 2:

$h(f(a), z)^{(I_1, E_1)} = 1$
(Rough work: $h(f(a), z)^{(I_1, E_1)} = h(f(1), 1)^{(I_1, E_1)} = h(2, 1)^{(I_1, E_1)} = 1$

$f(h(y, c))^{(I_1, E_1)} = 1$
(Rough work: $f(h(y, c))^{(I_1, E_1)} = f(h(3, 3))^{(I_1, E_1)} = f(3)^{(I_1, E_1)} = 1$

$P(h(f(a), z))^{(I_1, E_1)} = T$
(Rough work: $P(h(f(a), z))^{(I_1, E_1)} = P(1)^{(I_1, E_1)} = T$.)

$Q(y, h(a, b))^{(I_1, E_1)} = T$
(Rough work: $Q(3, h(1, 2))^{(I_1, E_1)} = Q(3, 1)^{(I_1, E_1)} = T$

Questions:

1. Give an interpretation I_4 and an environment E_4 such that $I_4 \not\models_{E_4} P(h(f(a), z))$

   The easiest way to make a predicate (like P) false is to define $P^{I_4} = \phi$.
   Let $I_4 = I_1$ except that $P^{I_4} = \phi$ and $E_4 = E_1$.
   The pair I_4 and E_4 make $P(h(f(a), z))$ false.

2. Give an interpretation I_5 and an environment E_5 such that $I_5 \not\models_{E_5} Q(y, h(a,b))$.

   Similar to 1, making $Q^{I_5} = \phi$ will do the trick.

Evaluating a predicate logic formula.

Finally, we are ready to evaluate formulas with free and bound variables with an interpretation and an environment.

Consider the formula

$$\alpha = (\exists x (\forall y \; R(x, y, h(z, c))))$$

① Give I_6 and E_6 such that $I_6 \vDash_{E_6} \alpha$.
② Give I_7 and E_7 such that $I_7 \nvDash_{E_7} \alpha$.

②. The truth value of α depends on the truth value of the predicate R. To make α false, it's enough to make R to always be false.

Let $I_7 = I_1$ with $R^{I_7} = \phi$.
Let $E_7 = E_1$
Thus, we have $I_7 \nvDash_{E_7} \alpha$. because R^{I_7} is false for any 3-tuple in D^3 $D = \{1, 2, 3\}$.

① To make α true, let's simplify α a bit under I_1 and E_1.
Let $I_6 = I_1$ and $E_6 = E_1$, Let $R^{I_6} = \phi$ for now

$$\alpha^{(I_6, E_6)} = (\exists x (\forall y \; R(x, y, h(1, 3))))^{(I_6, E_6)}$$
$$= (\exists x (\forall y \; R(x, y, 1)))^{(I_6, E_6)}$$

- To make α true, we need to choose one element of D as x, and make sure that every possible tuple $\langle x, y, 1 \rangle$ for every possible y in D is in R^{I_6}.
- Let's choose x to be 1. Then R^{I_6} has to contain the following 3 tuples $\langle 1,1,1 \rangle$, $\langle 1,2,1 \rangle$, $\langle 1,3,1 \rangle$.
- It suffices to define $R^{I_6} = \{\langle 1,1,1 \rangle, \langle 1,2,1 \rangle, \langle 1,3,1 \rangle\}$.