Semantic Entailment

Predicate Logic

Propositional Logic

\[ \Sigma \text{ semantically entails } c, \text{ denoted } \Sigma \models c, \iff I \models_E \Sigma \implies I \models_E c \text{ for any } \Sigma ^t = T \implies c^t = T \]

for any interpretation \( I \) and any environment \( E \).

A few notes for \( \Sigma \models c \) in predicate logic:

1. \( \Sigma \models c \) means if a pair of \( I \) and \( E \) makes every formula in \( \Sigma \) true, then \( I \) and \( E \) also make \( c \) true.

2. \( \Sigma \models c \) means \( \phi \models ((P_1 \land \ldots \land P_n) \rightarrow c) \)
   (if \( \Sigma = \{ P_1, P_2, \ldots, P_n \} \))

3. \( \Sigma \models c \) means \( ((P_1 \land \ldots \land P_n) \rightarrow c) \) is valid.

4. If \( \phi \models c \), then \( c \) is valid because \( I \models_E \phi \) for any \( I \) and \( E \).

Prove that \( \Sigma \models c \) holds.

1. Proof: Assume that there is an \( I \) and an \( E \) such that \( I \models_E \Sigma \).
   We need to show that \( I \models_E c \).

2. Proof by contradiction:
   Assume that there is an \( I \) and an \( E \) such that
   \( I \models_E \Sigma \) and \( I \not\models_E c \).
   We need to derive a contradiction.

Prove that \( \Sigma \not\models c \).

Give an \( I \) and an \( E \) such that \( I \models_E \Sigma \) and \( I \not\models_E c \).
Proving Semantic Entailment.

Q: Show that $(\exists y (\forall x P(x, y))) \models (\forall x (\exists y P(x, y)))$.

Proof by contradiction:

- Assume that there is an interpretation $I$ such that $(\exists y (\forall x P(x, y)))^I = T$ and $(\forall x (\exists y P(x, y)))^I = F$.
- Since $(\exists y (\forall x P(x, y)))^I = T$, there is an element $a \in \text{dom}(I)$ such that $(\forall x P(x, y))^{(I, E[y \mapsto a])} = T$.

If $(\forall x P(x, y))^{(I, E[y \mapsto a])} = T$, then $P(x, y)^{(I, E[x \mapsto b][y \mapsto a])} = T$ for every $d \in \text{dom}(I)$. ①

- Since $(\forall x (\exists y P(x, y)))^I = F$, we know that

$(- (\forall x (\exists y P(x, y)))^I = T$ or $(\exists y (\forall x (- P(x, y))))^I = T$.

This means that there is an element $b \in \text{dom}(I)$ such that $(\forall y (- P(x, y)))^{(I, E[x \mapsto b])} = T$.

If $(\forall y (- P(x, y)))^{(I, E[x \mapsto b])} = T$, then $(- P(x, y))^{(I, E[x \mapsto b][y \mapsto d])} = T$ for every $d \in \text{dom}(I)$. ②

- By equation ①, we know that $P(x, y)^{(I, E[x \mapsto b][y \mapsto a])} = T$ ③

- By equation ②, we know that $(- P(x, y))^{(I, E[x \mapsto b][y \mapsto a])} = T$. ④

- ③ and ④ mean that $P(x, y)$ is $T$ and $F$ under $I$ and $E[x \mapsto b][y \mapsto a]$. This is a contradiction.

QED
Proving Semantic Entailment:

Show that \( \phi \implies ((\forall x (x \rightarrow \beta)) \rightarrow ((\forall x \alpha) \rightarrow (\forall x \beta))) \)

Proof: We have no premise. So we need to show that

\( ((\forall x (x \rightarrow \beta)) \rightarrow ((\forall x \alpha) \rightarrow (\forall x \beta))) \) is valid.

Assume that there is an interpretation \( I \) and an environment \( E \) such that \( I \models E ((\forall x (x \rightarrow \beta)) \). We need to show that

\( I \models E ((\forall x \alpha) \rightarrow (\forall x \beta)) \). That means, we assume that

\( I \models E (\forall x \alpha) \) and we need to show that \( I \models E (\forall x \beta) \).

By definition of \( \forall \), \( I \models E (\forall x (x \rightarrow \beta)) \) means that

for every \( a \in \text{dom}(I) \), \( I \models E[\{x \rightarrow a\}] (x \rightarrow \beta) \).

By definition of \( \forall \), \( I \models E (\forall x \alpha) \) means that

for every \( a \in \text{dom}(I) \), \( I \models E[\{x \rightarrow a\}] \alpha \).

Thus, by definition of \( \rightarrow \), \( I \models E[\{x \rightarrow a\}] \beta \) for every \( a \in \text{dom}(I) \), which means \( I \models E (\forall \beta) \).

Q.E.D.

Proof by contradiction: Assume that there is an interpretation \( I \) and an environment \( E \) such that

\( I \not\models E ((\forall x (x \rightarrow \beta)) \rightarrow ((\forall x \alpha) \rightarrow (\forall x \beta))) \)

We need to derive a contradiction.

Our assumption means that \( I \models E (\forall x (x \rightarrow \beta)) \) and

\( I \not\models E ((\forall x \alpha) \rightarrow (\forall x \beta)) \). The latter means that

\( I \models E (\forall x \alpha) \) and \( I \not\models E (\forall x \beta) \).

By definition of \( \forall \), \( I \models E (\forall x (x \rightarrow \beta)) \) means that

for every \( a \in \text{dom}(I) \), \( I \models E[\{x \rightarrow a\}] (x \rightarrow \beta) \).

By definition of \( \forall \), \( I \models E (\forall x \alpha) \) means that

for every \( a \in \text{dom}(I) \), \( I \models E[\{x \rightarrow a\}] \alpha \).

By definition of \( \rightarrow \), we have that \( I \models E[\{x \rightarrow a\}] \beta \) for every \( a \in \text{dom}(I) \). This contradicts our assumption that

\( I \not\models E (\forall x \beta) \).

Q.E.D. ②
Disproving Semantic Entailment

Q: Show that \((\forall x (\exists y P(x,y))) \not\equiv (\exists y (\forall x P(x,y)))\)

Proof: We need to give an interpretation \(I\) such that

\((\forall x (\exists y P(x,y)))^I = T\) and \((\exists y (\forall x P(x,y)))^I = F\).

Define \(I\):

- **Domain**: \(\text{dom}(I) = \{1, 2, 3\}\)
- **Predicate**: \(P^I = \{<1,1>, <2,3>, <3,1>\}\)

Let's verify that the premise is true and the conclusion is false under \(I\). To do this, we need to evaluate \(P(x,y)\) under \(I\) and some environment \(E\).

Let \(E\) be an arbitrary environment.

To verify that \((\forall x (\exists y P(x,y)))^I\) is true, we need to verify that \((\exists y P(x,y))^\langle I,E[x\mapsto d]\rangle\) is true for every \(d \in \text{dom}(I)\).

**Case \([x\mapsto 1]\)**: \(P(x,y)^\langle I,E[x\mapsto 1][y\mapsto 1]\rangle = T\) because \(<1,1> \in P^I\).

Thus, \((\exists y P(x,y))^\langle I,E[x\mapsto 1]\rangle = T\).

**Case \([x\mapsto 2]\)**: \(P(x,y)^\langle I,E[x\mapsto 2][y\mapsto 3]\rangle = T\) because \(<2,3> \in P^I\).

Thus, \((\exists y P(x,y))^\langle I,E[x\mapsto 2]\rangle = T\).

**Case \([x\mapsto 3]\)**: \(P(x,y)^\langle I,E[x\mapsto 3][y\mapsto 1]\rangle = T\) because \(<3,1> \in P^I\).

Thus, \((\exists y P(x,y))^\langle I,E[x\mapsto 3]\rangle = T\).

Therefore, \((\forall x (\exists y P(x,y)))^I = T\).

continued on the next page...
Q: Show that \((\forall x (\exists y p(x,y))) \not \equiv (\exists y (\forall x p(x,y)))\).

Proof continued: We need to verify that \((\exists y (\forall x p(x,y))) = F\).

This is equivalent to verifying \((- (\exists y (\forall x p(x,y)))) = T\).

or \((\forall y (\exists x (\neg p(x,y)))) = T\).

We need to verify that:

\((\exists x (\neg p(x,y))) (I, E[y \mapsto d]) = T \text{ for every } d \in \text{dom}(I)\).

Case [y \mapsto 1]: \(p(x, y) (I, E[x \mapsto 2][y \mapsto 1]) = F\) because \(<2, 1> \notin \mathcal{P}\).

Thus, \((\exists x (\neg p(x, y))) (I, E[y \mapsto 1]) = T\).

Case [y \mapsto 2]: \(p(x, y) (I, E[x \mapsto 1][y \mapsto 2]) = F\) because \(<1, 2> \notin \mathcal{P}\).

Thus, \((\exists x (\neg p(x, y))) (I, E[y \mapsto 2]) = T\).

Case [y \mapsto 3]: \(p(x, y) (I, E[x \mapsto 1][y \mapsto 3]) = F\) because \(<1, 3> \notin \mathcal{P}\).

Thus, \((\exists x (\neg p(x, y))) (I, E[y \mapsto 3]) = T\).

Therefore, \((\exists y (\forall x p(x,y))) = F\).

QED
Disproving Semantic Entailment

Show that \((\forall x (\alpha) \Rightarrow (\forall x (\beta))) \neq (\forall x (\alpha \Rightarrow \beta))\)

(Thoughts: How can we come up with an interpretation \(I\), an environment \(E\), and formulas for \(\alpha\) and \(\beta\) such that
\(I = E ((\forall x (\alpha) \Rightarrow (\forall x (\beta)))\) and \(I \neq E (\forall x (\alpha \Rightarrow \beta))\)?

The easiest way to make \((\forall x (\alpha) \Rightarrow (\forall x (\beta)))\) true is to make \((\forall x (\alpha))\)
false, which means we need to make sure \(\alpha\) is false for at least
one element of the domain. Let \(\alpha\) be \(P(x)\) and let
\(P(x) \mid (I, E/x=a) = F\) for \(a \in \text{dom}(I)\).

To make \((\forall x (\alpha \Rightarrow \beta))\), the easiest way is to make sure there
exists \(b \in \text{dom}(I)\) such that \(\alpha (I, E/x=b) = T\) and \(\beta (I, E/x=b) = F\)
We could let \(\beta\) be \((\neg P(x))\), let \(\text{dom}(I) = \{a, b\}\) and
let \(P^1 = \{b\}\).

Proof: Let \(\alpha\) be \(P(x)\) and \(\beta\) be \((\neg P(x))\).
Let an interpretation \(I\) be: \(\text{dom}(I) = \{a, b\}\) and \(P^1 = \{b\}\).
Consider any environment \(E\).

We will show that \((I, E) (\forall x (\alpha) \Rightarrow (\forall x (\beta)))\) \(= T\) and \((I, E) (\forall x (\alpha \Rightarrow \beta))\) \(= F\).

1. \(a \in P\), so \(\alpha (I, E/x=a) = F\). Thus \((\forall x (\alpha)) (I, E) = F\).
   By definition of \(\Rightarrow\), \((\forall x (\alpha) \Rightarrow (\forall x (\beta))) (I, E) = T\).

2. \(b \in P\), so \(\alpha (I, E/x=b) = T\) and \(\beta (I, E/x=b) = F\).
   So \((\alpha \Rightarrow \beta) (I, E/x=b) = F\), or \((\forall x (\alpha \Rightarrow \beta)) (I, E) = F\).

Q.E.D.