Peano Arithmetic Nov 9.

Theorem : $(\forall x\ (x \times S(0) = x))$.

Proof by induction on x.

→ (Take the theorem, remove the quantifier and replace x by 0.)

Base case : we need to prove

$$0 \times S(0) = 0$$

Induction step: we need to prove.

$$(\forall x\ (\underbrace{(x \times S(0) = x)}_{\text{copy theorem} - \forall x} \rightarrow \underbrace{(S(x) \times S(0) = S(x))}_{\text{take (theorem} - \forall x\text{) and replace every } x \text{ by } S(x)}))$$

Write PA7 for this theorem.

$$((\underbrace{0 \times S(0) = 0}_{\text{base case}}) \rightarrow ((\underbrace{\forall x\ ((x \times S(0) = x) \rightarrow (S(x) \times S(0) = S(x)))}_{\text{induction step}}) \rightarrow (\underbrace{\forall x\ (x \times S(0) = x)}_{\text{theorem statement}})))$$

PA7: For each formula φ and variable x,

$$(\varphi[0/x] \rightarrow ((\forall x(\varphi[x/x] \rightarrow \varphi[S(x)/x])) \rightarrow (\forall x\ \varphi)))$$

solutions

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Theorem: $(\forall x (\forall y \; (x + y = y + x)))$

Proof by induction on x.

Base case: we need to prove

$(\; \forall y \; (0 + y = y + 0) \;)$

Induction step: we need to prove

$(\forall x \; ((\; \forall y \; (x + y = y + x) \;) \rightarrow (\; \forall y \; (S(x) + y = y + S(x)) \;)))$

Write PA7 for this theorem.

$$((\; \forall y \; (0 + y = y + 0) \;)$$
$$\rightarrow ((\forall x \; ((\forall y \; (x + y = y + x)) \rightarrow (\forall y \; (S(x) + y = y + S(x))))))$$
$$\rightarrow (\; \forall x \; (\forall y \; (x + y = y + x)) \;)))$$

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- We can define a predicate by using a formula.
- Given an interpretation I, a formula φ with free variables $x_1, \dots, x_k$ defines the k-ary predicate P_φ

$$P_\varphi = \{ \langle a_1, \dots, a_k \rangle \in dom(I)^k \mid \varphi^{(I, E[x_1 \mapsto a_1] \cdots [x_k \mapsto a_k])} = T \}$$

Examples :

① Define "$x \leq y$".

$\exists$

- $x \leq y$ iff the sum of x and some natural number z is y.

$\sim$ $x \leq y$ iff $(\exists z\ (x + z = y))$

② Define "$x < y$"

- $x < y$ iff the sum of x and some positive natural number z is y.
- $x < y$ iff $x \leq y$ and x is not equal to y.

$\sim$ $x < y$ iff $(\exists z\ (x + S(z) = y))$

$\sim$ $x < y$ iff $((x \leq y) \wedge (\neg (x = y)))$

③ Define "x is even".

- Even(x) iff x is equal to 2 times some natural number y.

$\sim$ Even(x) iff $(\exists y\ (x = y + y))$

$(\exists y\ (x = S(S(0)) \times y))$

④ Define "x is prime".

- Prime(x) iff x is greater than 1 and x ~~has~~ doesn't have a factor y that is greater than 1 and less than x.

$\sim$ Prime(x) iff

$$((S(0) < x) \wedge (\neg \underbrace{(\exists y\ (\exists z\ (((x = y \times z) \wedge (S(0) < y)) \wedge (y < x))))}_{\text{composite \#.}}))$$

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Theorem : $\leq$ is transitive.

$$(\forall x(\forall y(\forall z(((\underbrace{x\leq y}_{(\exists u(x+u=y))}) \land (\underbrace{y\leq z}_{(\exists v(y+v=z))})) \to (\overbrace{x\leq z}^{(\exists w(x+w=z))})))))$$

Proof : Hint: use Associativity of addition, EQsubs, EQtrans

Declare three fresh variables x_0, y_0, and z_0.

1	$(\exists u(x_0+u=y_0)) \land (\exists v(y_0+v=z_0))$	assumption
2	$(\exists u(x_0+u=y_0))$	$\land e$: 1
3	$(\exists v(y_0+v=z_0))$	$\land e$: 3
4	$x_0+u_0=y_0$, u_0 fresh	assumption
5	$y_0+v_0=z_0$, v_0 fresh	assumption.
6		$+\forall e \times 3$
7	$x_0+(u_0+v_0)=(x_0+u_0)+v_0$	Associativity of addition
8		
9	$(x_0+u_0)+v_0 = y_0+v_0$	EQsubs $(+v_0)$: 4.
10		
11	$y_0+v_0 = z_0$	reflexivity : 5.
12		
13	$x_0+(u_0+v_0)=z_0$	EQtrans(2) : 7, 9, 11.
14		
15	$(\exists w(x_0+w=z_0))$	$\exists i$: 13.
16	$(\exists w(x_0+w=z_0))$	$\exists e$: 3, 5–15
17	$(\exists w(x_0+w=z_0))$	$\exists e$: 2, 4–16

Apply $\to i$ and $\forall i$ 3 times.