EQtrans(k). $\dfrac{t_1 = t_2 \quad t_2 = t_3 \quad \cdots \quad t_k = t_{k+1}}{t_1 = t_{k+1}}$ for any terms $t_1, \ldots, t_{k+1}$.

"applying transitivity k times". Nov 6

Peano Arithmetic

EQsubs(r): $\dfrac{t_1 = t_2}{\underbrace{r[t_1/z]}_{\in D} \overset{*}{=} \underbrace{r[t_2/z]}_{\in D}}$ for any variable z, and terms r, t_1 and t_2

Axioms for Equality

EQ1: $(\forall x\,(x = x))$ (reflexivity)

EQ2: For each formula α and variable z,

$$(\forall x(\forall y\,((x = y) \to (\underbrace{\alpha[x/z]}_{=T} \overset{*}{\to} \underbrace{\alpha[y/z]}_{=T}))))$$

EQsymm: $(\forall x(\forall y\,((x = y) \to (y = x))))$ (symmetry)

EQtrans: $(\forall w\,(\forall x\,(\forall y\,((x = y) \to ((y = w) \to (x = w))))))$ (transitivity)

Peano Arithmetic:

Constant symbols: 0

Function symbols: $s^{(1)}$, $+^{(2)}$, $\times^{(2)}$

Interpretation I:

Domain: $\mathbb{N}$ the set of natural numbers $\{0, 1, 2, \ldots\}$

0^I = zero.

$s^{(1)}$: the successor function. $s(x) = x + 1$, $\forall x \in \mathbb{N}$.

$+^{(2)}$: the addition function.

$\times^{(2)}$: the multiplication function

Every natural number has a corresponding term:

0 : 0

1 : $s(0)$

2 : $s(s(0))$

3 : $s(s(s(0)))$

Peano Axioms: Nov 6

PA1: $(\forall x\ (\neg(s(x)=0)))$

- Every natural number's successor is not zero.
- Zero is not a successor.

PA2: $(\forall x(\forall y.\ ((s(x)=s(y))\rightarrow(x=y))))$

- If two numbers are the same, they must have the same predecessor. $P(x)$ $((x=y)\rightarrow(P(x)=P(y)))$
- No number has two different predecessors.

Two properties of addition:

PA3: $(\forall x\ (x+0=x))$

- The addition of any number and zero is the same number.
- Adding zero to any number yields the same number.

PA4: $(\forall x(\forall y.\ (x+s(y)=s(x+y))))$

- The addition of one x number and the successor of another y. number is the successor of the sum of the two numbers.
- Adding a successor yields the successor of adding the number.

Two properties of multiplication:

PA5: $(\forall x\ (x\times 0=0))$

- Multiplying any number by zero yields zero.

PA6: $(\forall x(\forall y.\ (x\times s(y)=(x\times y)+x)))$

- Multiplying one number and the successor of another number is the product of the two numbers plus the first number.

Peano Axioms. Nov. 6.

① Proving $(\forall x\, \varphi)$ by induction.

PA7: For each formula φ and variable x,

$$\underbrace{(\varphi[0/x]}_{\text{base case}} \rightarrow (\underbrace{(\forall x\,(\overbrace{\varphi[x/x]}^{\text{induction hypothesis}} \rightarrow \varphi[s(x)/x]))}_{\text{induction step}} \rightarrow \underbrace{(\forall x\, \varphi)}_{\text{conclusion}}))$$

This axiom allows us to prove properties of natural numbers by induction.

Base case: We need to prove $\varphi[0/x]$.

Induction step: We need to prove

$$(\forall x\,(\varphi[x/x] \rightarrow \varphi[s(x)/x]))$$

Consider an arbitrary natural number k.
Induction hypothesis: assume $\varphi[k/x]$ is true.
We need to prove $\varphi[s(k)/x]$ is true

...

By the principle of mathematical induction, $(\forall x\, \varphi)$ is true.

QED

② Proving $(\forall x\, \varphi)$ by $\forall i$ (direct proof).

Proof:

1	u fresh	assumption
⋮	⋮	
⋮	⋮	
n	$\varphi[u/x]$	
$n+1$	$(\forall x\, \varphi)$	$\forall i$: 1–n

Properties of Natural Numbers

Theorem: $(\forall x\,(\neg(S(x)=x)))$

Proof by induction:

Base case: We need to prove that $(\neg(S(0)=0))$

1	$(\forall x\,(\neg(S(x)=0)))$	PA1
2	$(\neg(S(0)=0))$	$\forall e$: 1 [0/x]

Induction step: We need to prove that

$\phi \vdash (\forall x\,((\neg(S(x)=x)) \to (\neg(S(S(x))=S(x)))))$

Hint: use $\neg i$ and PA2.

3	u fresh	assumption
4	$(\neg(S(u)=u))$	assumption
5	$(S(S(u))=S(u))$	assumption
6	$(\forall x\,(\forall y\,((S(x)=S(y)) \to (x=y))))$	PA2
7	$(S(S(u))=S(u)) \to (S(u)=u)$	
8	$(\forall y\,((S(S(u))=S(y)) \to (S(u)=y)))$	$\forall e$: 6 [S(u)/x]
9	$((S(S(u))=S(u)) \to (S(u)=u))$	$\forall e$: 8 [u/y]
10		
11		
12		
13	$(S(u)=u)$	$\to e$: 5, 9
14	$\perp$	$\perp i$: 4, 13
15	$(\neg(S(S(u))=S(u)))$	$\neg i$: 5–14
16	$((\neg(S(u)=u)) \to (\neg(S(S(u))=S(u))))$	$\to i$: 4–15
17	$(\forall x\,((\neg(S(x)=x)) \to (\neg(S(S(x))=S(x)))))$	$\forall i$: 3–16

Properties of Natural Numbers. Nov 6.

Theorem: $(\forall x\ (\neg (S(x) = x)))$.

Proof by induction continued:

Let's finish the proof.

15 $((\neg (S(0) = 0)) \rightarrow ((\forall x\ ((\neg (S(x) = x)) \rightarrow (\neg (S(S(x)) = S(x))))) \rightarrow (\forall x\ (\neg (S(x) = x)))))$ PA7: $[\varphi = (\neg (S(x) = x))]$

16 $((\forall x\ ((\neg (S(x) = x)) \rightarrow (\neg (S(S(x)) = S(x))))) \rightarrow (\forall x\ (\neg (S(x) = x))))$ $\rightarrow e$: 2, 15

17 $(\forall x (\neg (S(x) = x)))$ $\rightarrow e$: 14, 16.

QED

Theorem: $\phi \vdash (\forall x (\forall y (x + y = y + x)))$.
"Addition is commutative."

$\boxed{\text{Proof by induction on } x.}$

(x) Base case: We need to prove that
$(\forall y (0 + y = y + 0))$.

$\boxed{\text{We prove this by induction on } y.}$

(y) Base case: We need to prove that $(0+0 = 0+0)$.

1 $(\forall x (x = x))$. EQ1.
2 $(0+0 = 0+0)$ $\forall e$: 1 $[(0+0)/x]$

(y) Induction step: We need to prove that
$(\forall y ((0+y = y+0) \rightarrow (0 + S(y) = S(y) + 0)))$

Hint: use PA4, PA3 and EQsubs(r), EQsymm,

3	u fresh	assumption
4	$(0+u = u+0)$	assumption.
5		
6	$(0 + S(u) = S(0+u))$.	PA4 + $\forall e \times 2$.
7	$(S(0+u) = S(u+0))$.	EQsubs($S(x)$): 4.
8	$(u + 0 = u)$.	PA3 + $\forall e$.
9	$(S(u+0) = S(u))$.	EQsubs($S(x)$): 8.
10	$(S(u) + 0 = S(u))$	PA3 + $\forall e$. $+\forall e \times 2$
11	$((S(u)+0 = S(u)) \rightarrow (S(u) = S(u)+0))$	EQsymm
12	$(S(u) = S(u) + 0)$.	$\rightarrow e$: 10, 11
13		
14		
15	$(0 + S(u) = S(u) + 0)$	EQtrans(3): 6, 7, 9, 12
16	$((0 + u = u+0) \rightarrow (0 + S(u) = S(u)+0))$	$\rightarrow i$: 4–15
17	$(\forall y ((0+y = y+0) \rightarrow (0 + S(y) = S(y)+0)))$.	$\forall i$: 3–16

Thought process of proof:

$(0 + u = u + 0)$ assumption.

6 $0 + s(u) = s(0+u)$ PA4.

7 $s(0+u) = s(u+0)$ assumption + EQsubs.

9 $s(u+0) = s(u)$ PA3 + EQsubs.

12 $s(u) = s(u)+0.$ PA3 + EQsymm

$= s(u)+0$

15 $0 + s(u) = s(u) + 0$ EQtrans.

Proof by induction on x continued.

Base case: We need to prove that $(\forall y\,(0+y=y+0))$.

18 $((0+0=0+0)\to((\forall y\,(0+y=y+0)\to(0+S(y)=S(y)+0))$
$\to(\forall y\,(0+y=y+0))))$ PA7.

19 $((\forall y\,(0+y=y+0)\to(0+S(y)=S(y)+0))$
$\to(\forall y\,(0+y=y+0)))$ $\to e$: 2, 18

20 $(\forall y\,(0+y=y+0))$ $\to e$: 17, 19.

This completes the proof of the base case.

(x) Induction step: We need to prove that
$(\forall x\,(\forall y\,(x+y=y+x))\to(\forall y\,(S(x)+y=y+S(x))))$.

Consider an arbitrary $x\in\mathbb{N}$. We will prove that
$((\forall y\,(x+y=y+x))\to(\forall y\,(S(x)+y=y+S(x))))$.

Assume that $(\forall y\,(x+y=y+x))$. ⊛
We will prove that $(\forall y\,(S(x)+y=y+S(x)))$.
We prove this by induction on y.

(y) Base case: We will prove that $S(x)+0=0+S(x)$.

21 $(\forall y\,(0+y=y+0))$ by the base case of induction on x.

22 $(0+S(x)=S(x)+0)$ $\forall e$: 21

23

24 $((0+S(x)=S(x)+0)\to(S(x)+0=0+S(x)))$ EQsymm + $\forall e\times 2$

25 $(S(x)+0=0+S(x))$ $\to e$: 22, 24.

Properties of Natural Numbers. Nov 6

Proof by induction on x continued:

Induction step continued:

Induction on y continued:

(y) Induction step: We will prove that
$$(\forall y\, ((S(x)+y = y+S(x)) \rightarrow (S(x)+S(y) = S(y)+S(x)))).$$

Choose a fresh variable u.

We will prove $((S(x)+u = u+S(x)) \rightarrow (S(x)+S(u) = S(u)+S(x)))$.

Assume $(S(x)+u = u+S(x))$. (**)

We will prove $(S(x)+S(u) = S(u)+S(x))$.

30 $S(x) + S(u) = S(S(x)+u)$. PA4 + $\forall$e $\times 2$.

31 $S(x) + u = u + S(x)$ reflexivity on (**).

32 $S(S(x)+u) = S(u+S(x))$ EQsubs$(S(x))$: 31.

33 $S(u+S(x)) = S(S(u+x))$ PA4 + $\forall$e $\times 2$.

34 $x+u = u+x$. (*) + $\forall$e.

35 $((x+u = u+x) \rightarrow (u+x = x+u))$ EQsymm + $\forall$e $\times 2$.

36 $u+x = x+u$. $\rightarrow$e: 34, 35.

37 $S(S(u+x)) = S(S(x+u))$ EQsubs$(S(S(x)))$: 36.

38 $x+S(u) = S(x+u)$ PA4 + $\forall$e $\times 2$

39 $((x+S(u) = S(x+u)) \rightarrow (S(x+u) = x+S(u)))$ EQsymm + $\forall$e $\times 2$

40 $S(x+u) = x+S(u)$ $\rightarrow$e: 38, 39.

41 $S(S(x+u)) = S(x+S(u))$ EQsubs$(S(x))$: 40.

42 $x+S(u) = S(u)+x$ (*) + $\forall$e.

43 $S(x+S(u)) = S(S(u)+x)$ EQsubs$(S(x))$: 42.

44 $S(u)+S(x) = S(S(u)+x)$ PA4 + $\forall$e $\times 2$

45 $(S(u)+S(x) = S(S(u)+x)) \rightarrow (S(S(u)+x) = S(u)+S(x))$. EQsymm + $\forall$ex.

46 $S(S(u)+x) = S(u)+S(x)$ $\rightarrow$e: 44, 45.

47 $(S(x)+S(u) = S(u)+S(x))$ EQtrans(6): 30, 32, 33, 37, 41, 43, 46

Thought process of the proof:

$$(\forall y\,(x+y = y+x)) \quad (*)$$

$$S(x) + u = u + S(x) \quad (**)$$

30 $S(x) + S(u) = S(S(x) + u)$ PA4

32 $S(S(x)+u) = S(u+S(x))$ $(**)$ + EQsubs

33 $S(u+S(x)) = S(S+u(x))$ PA4 + EQsubs.

37 $S(S(u+x)) = S(S(x+u))$ $(*)$ + EQsubs.

41 $S(S(x+u)) = S(x+S(u))$ PA4 + EQsubs + EQsymm.

43 $S(x+S(u)) = S(S(u)+x)$ $(*)$ + EQsubs.

46 $S(S(u)+x) = S(u)+S(x)$ PA4 + EQsymm

47 $S(x)+S(u) = S(u)+S(x)$ EQtrans( ).