A4Q5(a). If $\{\alpha\} \not\models \beta$, then $\phi \nvdash (\alpha \to \beta)$.

Two approaches:

(a) $\{\alpha\} \not\models \beta \Rightarrow \phi \not\models (\alpha \to \beta) \Rightarrow \phi \nvdash (\alpha \to \beta)$.

(b) $\{\alpha\} \not\models \beta \Rightarrow \{\alpha\} \nvdash \beta \Rightarrow \phi \nvdash (\alpha \to \beta)$.

Approach (a):

Proof: Assume $\{\alpha\} \not\models \beta$

There is a truth valuation t such that $\alpha^t = T$ and $\beta^t = F$.

By definition of $\to$, $(\alpha \to \beta)^t = F$.

☆ $\phi^t = T$ because any truth valuation satisfies ϕ, and

☆ $(\alpha \to \beta)^t = F$. Therefore $\phi \not\models (\alpha \to \beta)$.

By the contrapositive of soundness, if $\phi \not\models (\alpha \to \beta)$,

then $\phi \nvdash (\alpha \to \beta)$.

QED.

A4Q5(a) If $\{\alpha\} \not\models \beta$, then $\phi \nvdash (\alpha \to \beta)$.

Approach (b)

Proof: Assume $\{\alpha\} \not\models \beta$.

By the contrapositive of soundness of natural deduction,

if $\{\alpha\} \not\models \beta$, then $\{\alpha\} \nvdash \beta$.

Now we need to show that if $\{\alpha\} \nvdash \beta$, then $\phi \nvdash (\alpha \to \beta)$.
We prove its contrapositive, that is,

if $\phi \vdash (\alpha \to \beta)$ then $\{\alpha\} \vdash \beta$.

Since $\phi \vdash (\alpha \to \beta)$. There is a natural deduction proof as shown below.

1	α	assumption
⋮	⋮	
n	β	
$n+1$	$(\alpha \to \beta)$	$\to i$: $1-n$.

From this, we can construct a new natural deduction proof.

1	α	premise
⋮	⋮	
n	β	

This proof shows that $\{\alpha\} \vdash \beta$.

QED