Proving Undecidability via Reductions

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Lecture 24

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Outline

Proving Undecidability via Reductions
Learning Goals
Revisiting the Learning Goals
Learning Goals

By the end of this lecture, you should be able to:

Proving that a problem is undecidable by a reduction from the halting problem

Define reduction.

Describe at a high level how we can use reduction to prove that a decision problem is undecidable.

Prove that a decision problem is undecidable by using a reduction from the halting problem.
Proving that other problems are undecidable

We proved that the Halting problem is undecidable. How do we prove that another problem is undecidable?

- We could prove it from scratch, or
- We could prove that it is as difficult as the halting problem. Hence, it must be undecidable.
We will prove undecidability via reductions. Problem A is reducible to problem B.

- Given an algorithm for solving B, we could use it to solve A.
- If B is decidable, then A is decidable.
- If A is undecidable, then B is undecidable.
Theorem: Problem \( X \) is undecidable.

Proof by Contradiction.

Assume that there is an algorithm \( B \), which solves problem \( X \).

We will construct algorithm \( A \), which uses algorithm \( B \) to solve the halting problem. (Describe algorithm \( A \).)

Since algorithm \( B \) solves problem \( X \), algorithm \( A \) solves the halting problem.

This contradicts with the fact that the halting problem is undecidable. Therefore, algorithm \( B \) does not exist.
Example 1 of reduction proofs

The halting-no-input problem:

*Given a program $P$ which takes no input, does $P$ halt?*

Theorem: The halting-no-input problem is undecidable.
The both-halt problem:

*Given two programs $P_1$ and $P_2$ which take no input, do both programs halt?*

Theorem: The both-halt problem is undecidable.
CQ 1: To prove that the both-halt problem is undecidable, does the following reduction work?

Let $P_1$ run $P$ with input $I$. Let $P_2$ do nothing and terminate immediately.

(A) Yes
(B) No
(C) I don’t know.
CQ 2 Does this reduction work?

CQ 2: To prove that the both-halt problem is undecidable, does the following reduction work?

Let $P_1$ run $P$ with input $I$. Let $P_2$ run an infinite loop and never terminate.

(A) Yes
(B) No
(C) I don’t know.
Example 3 of reduction proofs

The exists-halting-input problem

*Given a program $P$, does there exist an input $I$ such that $P$ halts with input $I$?*

Theorem The exists-halting-input problem is undecidable.
Example 4 of reduction proofs

The partial-correctness problem

*Given a Hoare triple* `(|P| C|Q|)`,

*is the triple satisfied under partial correctness?*

**Theorem:** The partial-correctness problem is undecidable.
Revisiting the learning goals

By the end of this lecture, you should be able to:
Proving that a problem is undecidable by a reduction from the halting problem

- Define reduction.
- Describe at a high level how we can use reduction to prove that a decision problem is undecidable.
- Prove that a decision problem is undecidable by using a reduction from the halting problem.