Proving Undecidability via Reductions

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Lecture 23

Outline

Learning Goals

By the end of this lecture, you should be able to:

- Define reduction.
- Describe at a high level how we can use reduction to prove that a decision problem is undecidable.
- Prove that a decision problem is undecidable by using a reduction from the halting problem.

Outline

Proving that other problems are undecidable

We proved that the halting problem is undecidable.

How do we prove that another problem is undecidable?

- We could prove it from scratch, or
- We could prove that it is as difficult as the halting problem. Hence, it must be undecidable.

Proving undecidability via reductions

We will prove undecidability via reductions.

Reduce the halting problem to problem P_B.

- Given an algorithm for solving P_B,
  we could use it to solve the halting problem.
- If P_B is decidable, then the halting problem is decidable.
- If the halting problem is undecidable, then P_B is undecidable.

Proving undecidability via reductions

Theorem: Problem P_B is undecidable.

Proof by Contradiction.

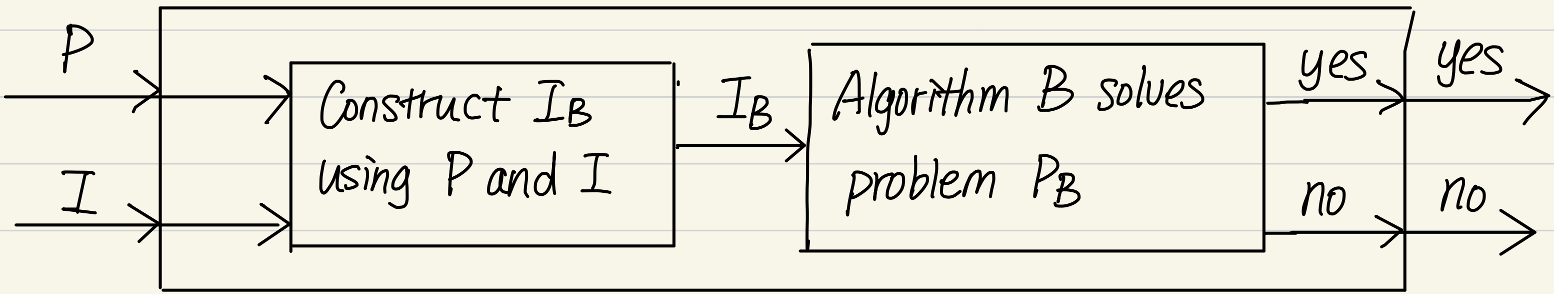
Assume that there is an algorithm B, which solves problem P_B.

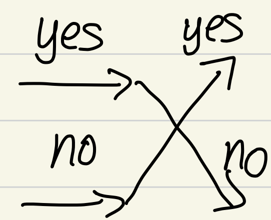
We will construct algorithm A, which uses algorithm B to solve the halting problem. (Describe algorithm A.)

Since algorithm B solves problem P_B, algorithm A solves the halting problem, which contradicts with the fact that the halting problem is undecidable.

Therefore, problem P_B is undecidable. $\square$

Algorithm A solves the halting problem





Outline

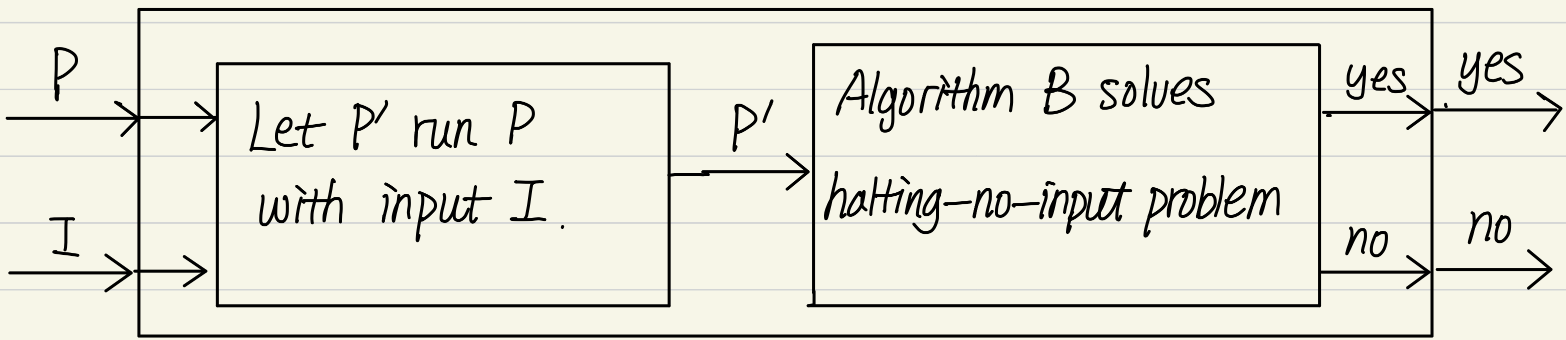
Example 1 of reduction proofs

The halting-no-input problem:

Given a program P which takes no input, does P halt?

Theorem: The halting-no-input problem is undecidable.

Algorithm A solves the halting problem




```
P'() {
    return P(I);
}
```

Proof by contradiction:

Assume that there is an algorithm B, which solves the halting-no-input problem for any program P.

$\forall$

We will construct an algorithm A to solve the halting problem. Algorithm A works as follows:

- A takes two inputs, a program P and an input I.
- Let program P' run P with input I and output the result of P(I).
- Run algorithm B with P' as the input.
- Return the result of B(P').

By our construction of algorithm A,

P′ halts if and only if P halts on input I.

Since algorithm B solves the halting-no-input problem, algorithm A solves the halting problem, which contradicts the fact that the halting problem is undecidable.

Therefore, the halting-no-input problem is undecidable.

$\blacksquare$

Example 2 of reduction proofs

The both-halt problem:

Given two programs P_1 *and* P_2 *which take no input, do both programs halt?*

Theorem: The both-halt problem is undecidable.

Example 3 of reduction proofs

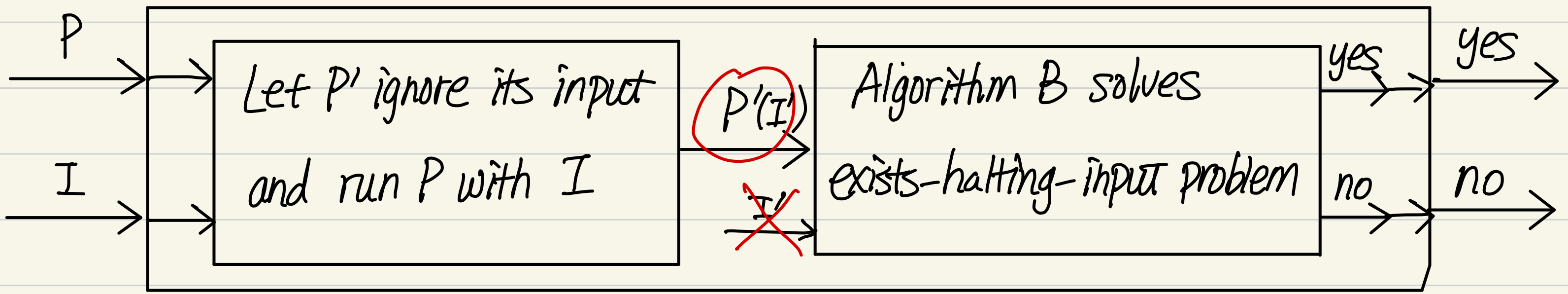
which takes an input

The exists-halting-input problem

> *Given a program P, does there exist an input I such that P halts with input I?*

Theorem The exists-halting-input problem is undecidable.

Algorithm A solves the halting problem



input to program P'

input to algorithm B.

```
" P'(I') {
    return P(I);
} "
```

Proof by contradiction:

Assume that there is an algorithm B, which solves the exists-halting-input problem for any program P.

We will construct algorithm A to solve the halting problem.
Algorithm A works as follows:

- A takes two inputs, a program P and an input I.
- Let program P' ignore its input, run P with input I and return P(I);

  common idea.
- Run algorithm B with P' as its input.
- Return B(P').

By our construction of algorithm A,

P halts on input I if and only if there exists an input I' such that P' halts on input I'.

Since B solves the exists-halting-input problem, then A solves the halting problem, which contradicts the fact that the halting problem is undecidable.

Therefore, the exists-halting-input problem is undecidable.

$\blacksquare$

Revisiting the learning goals

By the end of this lecture, you should be able to:

- Define reduction.
- Describe at a high level how we can use reduction to prove that a decision problem is undecidable.
- Prove that a decision problem is undecidable by using a reduction from the halting problem.