Program Verification
While Loops

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Lecture 20

Based on work by J. Buss, L. Kari, A. Lubiw, B. Bonakdarpour, D. Maftuleac, C. Roberts, R. Trefler, and P. Van Beek
Outline

Program Verification: While Loops
Learning Goals
Proving Partial Correctness - Example 1
Proving Partial Correctness - Example 2
Proving Termination
Revisiting the Learning Goals
Learning Goals

By the end of this lecture, you should be able to:
Partial correctness for while loops
   ▶ Determine whether a given formula is an invariant for a while loop.
   ▶ Find an invariant for a given while loop.
   ▶ Prove that a Hoare triple is satisfied under partial correctness for a program containing while loops.

Total correctness for while loops
   ▶ Determine whether a given formula is a variant for a while loop.
   ▶ Find a variant for a given while loop.
   ▶ Prove that a Hoare triple is satisfied under total correctness for a program containing while loops.
Proving Total Correctness of While Loops

- Partial correctness
- Termination
Proving Partial Correctness of While Loops

\[
\begin{align*}
(P) \\
(I) & \quad \text{implied (A)} \\
\textbf{while} \ (B) \ \{ \\
\ (I \land B) & \quad \text{partial-while} \\
C & \\
\ (I) & <\text{justify based on } C - \text{a subproof}> \\
\} \\
(I \land (\neg B)) & \quad \text{partial-while} \\
(Q) & \quad \text{implied (B)}
\end{align*}
\]

Proof of implied (A): \((P \rightarrow I)\)
Proof of implied (B): \(((I \land (\neg B)) \rightarrow Q)\)

\(I\) is called a loop invariant. We need to determine \(I\)!
What is a loop invariant?

A loop invariant is:

- A relationship among the variables. (A predicate formula involving the variables.)
- The word “invariant” means something that does not change.
- It is true before the loop begins.
- It is true at the start of every iteration of the loop and at the end of every iteration of the loop.
- It is true after the loop ends.
Proving partial correctness of while loops

Indicate the places in the program where the loop invariant is true.

\(| (x \geq 0) |\)
y = 1;
z = 0;
while (z != x) {
    z = z + 1;
    y = y * z;
}
\(| (y = x!) |\)
Proving partial correctness of a while loop

Steps to follow:
- Find a loop invariant.
- Complete the annotations.
- Prove any implied’s.

How do we find a loop invariant???
How do we find a loop invariant?

First, we need to understand the purpose of an invariant.

- The postcondition is the ultimate goal of our while loop.
- At every iteration, we are making progress towards the postcondition.
- The invariant is describing the progress we are making at every iteration.
Example 1:
Prove that the following triple is satisfied under partial correctness.

\[ \langle (x \geq 0) \rangle \]

\[ y = 1; \]

\[ z = 0; \]

\[ \textbf{while} \ (z \neq x) \ {\} \]

\[ \quad z = z + 1; \]

\[ \quad y = y \ast z; \]

\[ \langle (y = x!) \rangle \]
Finding a loop invariant

Step 1: Write down the values of all the variables every time the while test is reached.

\( (x \geq 0) \)

```plaintext
y = 1;
z = 0;
while (z != x) {
    z = z + 1;
y = y * z;
}
( (y = x!) )
```
Finding a loop invariant

Step 2: Find relationships among the variables that are true for every while test. These are our candidate invariants. Come up with some invariants in the next 2 minutes.

\[
\begin{align*}
\text{l}((x \geq 0)) & \\
y &= 1; \\
z &= 0; \\
\textbf{while} (z \neq x) \{ \\
    & \quad z = z + 1; \\
    & \quad y = y * z; \\
\} \\
\text{l}(y = x!)
\end{align*}
\]

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CQ 1: Is \((\neg (z = x))\) a loop invariant?

(A) Yes  (B) No  (C) I don’t know...

\[
\begin{align*}
\langle x \geq 0 \rangle \\
y & = 1; \\
z & = 0; \\
\text{while} (z \neq x) \{ \\
    & z = z + 1; \\
    & y = y \times z; \\
\} \\
\langle y = x! \rangle
\end{align*}
\]

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<td>120 = 5!</td>
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CQ 2: Is \((z \leq x)\) a loop invariant?

(A) Yes (B) No (C) I don’t know...

\[
\begin{align*}
\{ (x \geq 0) \} \\
y &= 1; \\
z &= 0; \\
\textbf{while } (z \neq x) \{ \\
& \quad z = z + 1; \\
& \quad y = y \times z; \\
\} \\
\{ (y = x!) \}
\end{align*}
\]

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<td>120 = 5!</td>
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CQ 3: Is (y = z!) a loop invariant?

(A) Yes (B) No (C) I don’t know...

\(\{ (x \geq 0) \}\)

\begin{align*}
y &= 1; \\
z &= 0; \\
\textbf{while} \ (z \neq x) \ {\{ } \\
\quad & z = z + 1; \\
\quad & y = y \times z; \\
\} \end{align*}

\(\{ (y = x!) \}\)

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| 1 \!
| 1 \!
| 2 \!
| 6 \!
| 24 \!
| 120 \!|
CQ 4: Is \((y = x!\)) a loop invariant?

(A) Yes (B) No (C) I don’t know...

\[\left( (x \geq 0) \right) \]
\(y = 1;\)
\(z = 0;\)
\textbf{while} \ (z \neq x) \ {\}
\quad \{\text{\quad z = z + 1;\}
\quad \text{\quad y = y * z;\}
\}
\[\left( (y = x!) \right) \]
CQ 5: Is \(((z \leq x) \land (y = z!))\) a loop invariant?

(A) Yes (B) No (C) I don’t know...

\[
\begin{align*}
&\{ (x \geq 0) \} \\
&y = 1; \\
&z = 0; \\
&\textbf{while} (z \neq x) \{ \\
&\quad z = z + 1; \\
&\quad y = y \times z; \\
&\} \\
&\{ (y = x!) \}
\end{align*}
\]

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<td>120</td>
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Finding a loop invariant

Step 3: Try each candidate invariant until we find one that works for our proof.

\[ \lnot (x \geq 0) \]

\[ y = 1; \]
\[ z = 0; \]
\begin{verbatim}
while (z != x) {
    z = z + 1;
    y = y * z;
}
\end{verbatim}

\[ \lnot (y = x!) \]

\begin{tabular}{c|c|c}
\hline
x & z & y \\
\hline
5 & 0 & 1 = 0! \\
5 & 1 & 1 = 1! \\
5 & 2 & 2 = 2! \\
5 & 3 & 6 = 3! \\
5 & 4 & 24 = 4! \\
5 & 5 & 120 = 5! \\
\hline
\end{tabular}
How do we find an invariant?

A recap of the steps to find an invariant:

- Write down the values of all the variables every time the while test is reached.
- Find relationships among the variables that are true for every while test. These are our candidate invariants.
- Try each candidate invariant until we find one that works for our proof.
CQ 6: Which loop invariant would you like to try first?

(A) \( z \leq x \)

(B) \( y = z! \)

(C) \( ((z \leq x) \land (y = z!)) \)
Partial While - Example 1 ($(z \leq x)$ as the invariant)

\[
\begin{align*}
\mathcal{L}(x \geq 0) & \\
\mathcal{L}(0 \leq x) & \\
y = 1; & \text{assignment} \\
\mathcal{L}(0 \leq x) & \\
z = 0; & \text{assignment} \\
\mathcal{L}(z \leq x) & \\
\textbf{while} (z \neq x) \{ & \\
\quad \mathcal{L}(\((z \leq x) \land (\neg (z = x)))) & \text{partial–while} \\
\quad \mathcal{L}(z + 1 \leq x) & \text{implied (B)} \\
\quad z = z + 1; & \text{assignment} \\
\quad \mathcal{L}(z \leq x) & \\
y = y \ast z; & \text{assignment} \\
\quad \mathcal{L}(z \leq x) & \\
\} \\
\mathcal{L}(\((z \leq x) \land (\neg (\neg (z = x)))) & \text{partial–while} \\
\mathcal{L}(y = x!) & \text{implied (C)}
\end{align*}
\]
We used \((z \leq x)\) as the invariant.

CQ 7: Is there a proof for implied (A)?

\[ ((x \geq 0) \rightarrow (0 \leq x)) \]

(A) Yes
(B) No
(C) I don't know.
CQ 8 Is there a proof for implied (B)?

We used \((z \leq x)\) as the invariant.

**CQ 8:** Is there a proof for implied (B)?

\[
((((z \leq x) \land \neg(z = x))) \rightarrow (z + 1 \leq x))
\]

(A) Yes

(B) No

(C) I don’t know.
We used \((z \leq x)\) as the invariant.

\textbf{CQ 9:} Is there a proof for implied (C)?

\[ (((z \leq x) \land (\neg(\neg(z = x)))) \to (y = x!)) \]

(A) Yes

(B) No

(C) I don’t know.
Example 1: Summary of invariants

Which invariant leads to a valid proof?

- \((z \leq x)\) does NOT lead to a valid proof.
- \((y = z!)\) does lead to a valid proof.
- \(((z \leq x) \land (y = z!))\) does lead to a valid proof.
Example 2:
Prove that the following triple is satisfied under partial correctness.

\[
\{ (x \geq 0) \}
\]
y = 1;
z = 0;
while (z < x) {
    z = z + 1;
    y = y * z;
}
\{ (y = x!) \}

Let’s try using \((y = z!\) as the invariant in our proof.
CQ 10: For example 2, which invariant leads to a valid proof?

(A) \( z \leq x \)
(B) \( y = z! \)
(C) \( ((z \leq x) \land (y = z!)) \)
(D) Two of (A), (B), and (C).
(E) All of (A), (B), and (C).
Which invariant leads to a valid proof?

To check whether an invariant leads to a valid proof, we need to check whether all of the implied’s can be proved.
CQ 11 Is there a proof for implied (A)?

We used \((y = z!)\) as the invariant.

**CQ 11:** Is there a proof for implied (A)?

\[ ((x \geq 0) \rightarrow (1 = 0!)) \]

(A) Yes

(B) No

(C) I don’t know.
CQ 12: Is there a proof for implied (B)?

We used \((y = z!)\) as the invariant.

**CQ 12:** Is there a proof for implied (B)?

\[(((y = z!) \land (z < x)) \rightarrow (y \times (z + 1) = (z + 1)!))\]

(A) Yes
(B) No
(C) I don’t know.
CQ 13 Is there a proof for implied (C)?

We used \((y = z!\)) as the invariant.

**CQ 13:** Is there a proof for implied (C)?

\[(((y = z!) \land \neg(z < x))) \rightarrow (y = x!)\]

(A) Yes  
(B) No  
(C) I don’t know.
CQ 14 Is there a proof for implied (C)?

We used \(((y = z!) \land (z \leq x))\) as the invariant.

CQ 14: Is there a proof for implied (C)?

\[(((y = z!) \land (z \leq x)) \land (\neg(z < x))) \rightarrow (y = x!))\]

(A) Yes
(B) No
(C) I don’t know.
CQ 15: For example 2, which invariant leads to a valid proof?

(A) \((z \leq x)\)
(B) \((y = z!)\)
(C) \(((z \leq x) \land (y = z!))\)
(D) Two of (A), (B), and (C).
(E) All of (A), (B), and (C).
Example 2: Summary of invariants

Which invariant leads to a valid proof?

- \((z \leq x)\) does NOT lead to a valid proof.
- \((y = z!)\) does NOT lead to a valid proof.
- \(((z \leq x) \land (y = z!))\) does lead to a valid proof.
Find an integer expression that

- is non-negative before the loop starts, at every iteration of the loop, and after the loop ends.
- decreases by at least 1 at every iteration of the loop.

This integer expression is called a variant (something that changes).

The loop must terminate because a non-negative integer can decrease by 1 a finite number of times.
Example 2: Finding a variant

Example 2:
Prove that the following program terminates.

\begin{verbatim}
y = 1;
z = 0;
while (z < x) {
    z = z + 1;
y = y * z;
}
\end{verbatim}

How do we find a variant? The loop guard \((z < x)\) helps.
Example 2: Proof of Termination

Consider the variant \((x - z)\).

Before the loop starts, \((x - z) \geq 0\) because the precondition is \((x \geq 0)\) and the second assignment mutates \(z\) to be 0.

During every iteration of the loop, \((x - z)\) decreases by 1 because \(x\) does not change and \(z\) increases by 1.

Thus, \(x - z\) will eventually reach 0.

When \(x - z = 0\), the loop guard \(z < x\) will terminate the loop.
Revisiting the learning goals

By the end of this lecture, you should be able to:

Partial correctness for while loops

- Determine whether a given formula is an invariant for a while loop.
- Find an invariant for a given while loop.
- Prove that a Hoare triple is satisfied under partial correctness for a program containing while loops.

Total correctness for while loops

- Determine whether a given formula is a variant for a while loop.
- Find a variant for a given while loop.
- Prove that a Hoare triple is satisfied under total correctness for a program containing while loops.