Program Verification
While Loops

Alice Gao
Lecture 20

Based on work by J. Buss, L. Kari, A. Lubiw, B. Bonakdarpour, D. Maftuleac, C. Roberts, R. Trefler, and P. Van Beek
Program Verification: While Loops

Learning Goals

Proving Partial Correctness - Example 1
Proving Partial Correctness - Example 2
Proving Termination
Revisiting the Learning Goals
Learning Goals

By the end of this lecture, you should be able to:
Partial correctness for while loops
- Determine whether a given formula is an invariant for a while loop.
- Find an invariant for a given while loop.
- Prove that a Hoare triple is satisfied under partial correctness for a program containing while loops.

Total correctness for while loops
- Determine whether a given formula is a variant for a while loop.
- Find a variant for a given while loop.
- Prove that a Hoare triple is satisfied under total correctness for a program containing while loops.
Proving Total Correctness of While Loops

- Partial correctness
- Termination
Proving Partial Correctness of While Loops

\[
\{P\} \quad I \quad \text{implied (A)} \\
\textbf{while (B)} \{ \\
\quad \{I \land B\} \quad \text{partial-while} \\
\quad C \\
\quad \{I\} \quad <\text{justify based on C – a subproof}> \\
\} \\
\{I \land (\neg B)\} \quad \text{partial-while} \\
\{Q\} \quad \text{implied (B)}
\]

Proof of implied (A): \(P \rightarrow I\)
Proof of implied (B): \(((I \land (\neg B)) \rightarrow Q)\)

\(I\) is called a loop invariant. We need to determine \(I!\)
What is a loop invariant?

A loop invariant is:

- A relationship among the variables. (A predicate formula involving the variables.)
- The word “invariant” means something that does not change.
- It is true before the loop begins.
- It is true at the start of every iteration of the loop and at the end of every iteration of the loop.
- It is true after the loop ends.
Indicate the places in the program where the loop invariant is true.

\(\{ (x \geq 0) \}\)

\(y = 1;\)

\(z = 0;\)

\textbf{while}  \( (z \neq x) \)  \{ 
\(z = z + 1;\)

\(y = y * z;\)
\}

\(\{ (y = x!) \}\)
Steps to follow:

- Find a loop invariant.
- Complete the annotations.
- Prove any implied’s.

How do we find a loop invariant???
How do we find a loop invariant?

First, we need to understand the purpose of an invariant.

- The postcondition is the ultimate goal of our while loop.
- At every iteration, we are making progress towards the postcondition.
- The invariant is describing the progress we are making at every iteration.
**Example 1:**
Prove that the following triple is satisfied under partial correctness.

\[ \langle x \geq 0 \rangle \]

\[ y = 1; \]
\[ z = 0; \]
\[ \textbf{while} \ (z \neq x) \ { \}
\[ \quad z = z + 1; \]
\[ \quad y = y \ast z; \]
\[ \} \]
\[ \langle y = x! \rangle \]
Finding a loop invariant

Step 1: Write down the values of all the variables every time the while test is reached.

\( (x \geq 0) \)

\( y = 1; \)
\( z = 0; \)
\( \textbf{while } (z != x) \{ \)
\( 		z = z + 1; \)
\( 	ty = y * z; \)
\( \} \)
\( ((y = x!)) \)
Finding a loop invariant

Step 2: Find relationships among the variables that are true for every while test. These are our candidate invariants.

Come up with some invariants in the next 2 minutes.

\[
\begin{align*}
\| (x \geq 0) \|
\end{align*}
\]

\[
\begin{align*}
y & = 1; \\
z & = 0; \\
\textbf{while } (z \neq x) \{ \\
& \quad z = z + 1; \\
& \quad y = y \times z; \\
\} \\
\| (y = x!) \|
\end{align*}
\]

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CQ 1: Is \( (z = x) \) a loop invariant?

(A) Yes  (B) No  (C) I don’t know...

\[
\begin{align*}
\{ (x \geq 0) \} & \\
y &= 1; \\
z &= 0; \\
\textbf{while} \ (z \neq x) \ { \\
  z &= z + 1; \\
  y &= y \times z; \\
}\} \\
\{ (y = x!) \}
\end{align*}
\]

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CQ 2: Is (z ≤ x) a loop invariant?

(A) Yes (B) No (C) I don’t know...

\(\{ (x \geq 0) \}\)

\[\begin{align*}
y &= 1; \\
z &= 0; \\
\textbf{while} \ (z \neq x) \{ \\
&\quad z = z + 1; \\
&\quad y = y \times z; \\
\} \\
\{ (y = x!) \}\]

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CQ 3: Is \((y = z!)\) a loop invariant?

(A) Yes (B) No (C) I don’t know...

\[
\begin{align*}
\{ & (x \geq 0) \} \\
y &= 1; \\
z &= 0; \\
\textbf{while} \ (z \neq x) \ \{ \\
\quad & z = z + 1; \\
\quad & y = y \ast z; \\
\} \\
\{ & (y = x!) \}
\end{align*}
\]

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CQ 4: Is \((y = x!)\) a loop invariant?

(A) Yes (B) No (C) I don’t know...

\(\quad \\| (x \geq 0) \|\)
\[
\begin{align*}
y & = 1; \\
z & = 0; \\
\textbf{while} & \quad (z \neq x) \; \{ \\
& \quad z = z + 1; \\
& \quad y = y \times z; \\
& \} \\
\quad \quad \quad \| (y = x!) \|
\end{align*}
\]

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CQ 5: Is \(((z \leq x) \land (y = z!))\) a loop invariant?

(A) Yes (B) No (C) I don’t know...

\[
\begin{align*}
\| (x \geq 0) \| \\
y &= 1; \\
z &= 0; \\
\textbf{while} (z \neq x) \{ \\
\quad z &= z + 1; \\
\quad y &= y \times z; \\
\} \\
\| (y = x!) \|
\end{align*}
\]

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Finding a loop invariant

Step 3: Try each candidate invariant until we find one that works for our proof.

\((x \geq 0)\)
\begin{verbatim}
y = 1;
z = 0;
while (z != x) {
    z = z + 1;
y = y * z;
}
\end{verbatim}

\((y = x!))

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How do we find an invariant?

A recap of the steps to find an invariant:

- Write down the values of all the variables every time the while test is reached.
- Find relationships among the variables that are true for every while test. These are our candidate invariants.
- Try each candidate invariant until we find one that works for our proof.
CQ 6: Which loop invariant would you like to try first?

(A) \( z \leq x \)
(B) \( y = z! \)
(C) \( ((z \leq x) \land (y = z!)) \)
Partial While - Example 1 \(((z \leq x) \text{ as the invariant})\)

\[
\begin{aligned}
\mathcal{L}(\mathbf{x} \geq 0) &
\mathcal{L}(0 \leq \mathbf{x}) \\
y & = 1; \\
\mathcal{L}(0 \leq \mathbf{x}) &
\mathcal{L}(z = 0); \\
\mathcal{L}(z \leq \mathbf{x})&
\text{assignment} \\
\text{while} \ (z \neq \mathbf{x}) \ {\{ \\
\mathcal{L}(((z \leq \mathbf{x}) \land (\neg(z = \mathbf{x})))) &
\mathcal{L}(z + 1 \leq \mathbf{x}) \\
\mathcal{L}(z = z + 1); \\
\mathcal{L}(z \leq \mathbf{x}) &
\text{assignment} \\
y & = y \ast z; \\
\mathcal{L}(z \leq \mathbf{x}) &
\text{assignment} \\
\} \\
\mathcal{L}(((z \leq \mathbf{x}) \land (\neg(\neg(z = \mathbf{x})))) &
\mathcal{L}(y = x!)) \\
\mathcal{L}(y = x!)) &
\text{partial-while} \\
\mathcal{L}(z \leq \mathbf{x}) &
\text{implied (A)} \\
\mathcal{L}(z + 1 \leq \mathbf{x}) &
\text{implied (B)} \\
\mathcal{L}(y = x!) &
\text{implied (C)}
\end{aligned}
\]
CQ 7 Is there a proof for implied (A)?

We used \((z \leq x)\) as the invariant.

**CQ 7:** Is there a proof for implied (A)?

\[((x \geq 0) \rightarrow (0 \leq x))\]

(A) Yes  
(B) No  
(C) I don’t know.
CQ 8: Is there a proof for implied (B)?

We used \((z \leq x)\) as the invariant.

**CQ 8:** Is there a proof for implied (B)?

\[
(((z \leq x) \land \lnot(z = x))) \rightarrow (z + 1 \leq x)
\]

(A) Yes

(B) No

(C) I don't know.
CQ 9 Is there a proof for implied (C)?

We used \((z \leq x)\) as the invariant.

**CQ 9:** Is there a proof for implied (C)?

\[((((z \leq x) \land (\neg(\neg(z = x)))))) \rightarrow (y = x!))\]

(A) Yes
(B) No
(C) I don’t know.
Example 1: Summary of invariants

Which invariant leads to a valid proof?

- \((z \leq x)\) does NOT lead to a valid proof.
- \((y = z!)\) does lead to a valid proof.
- \(((z \leq x) \land (y = z!))\) does lead to a valid proof.
Example 2:
Prove that the following triple is satisfied under partial correctness.

\[ \langle (x \geq 0) \rangle \]
\[ y = 1; \]
\[ z = 0; \]
\[ \textbf{while} \quad (z < x) \quad \{ \]
\[ \quad z = z + 1; \]
\[ \quad y = y \ast z; \]
\[ \} \]
\[ \langle (y = x!) \rangle \]

Let’s try using \((y = z!))\ as the invariant in our proof.
CQ 10: For example 2, which invariant leads to a valid proof?

(A) \((z \leq x)\)

(B) \((y = z!)\)

(C) \(((z \leq x) \land (y = z!))\)

(D) Two of (A), (B), and (C).

(E) All of (A), (B), and (C).
Which invariant leads to a valid proof?

To check whether an invariant leads to a valid proof, we need to check whether all of the implied’s can be proved.
CQ 11 Is there a proof for implied (A)?

We used \((y = z!)\) as the invariant.

CQ 11: Is there a proof for implied (A)?

\[ (((x \geq 0) \rightarrow (1 = 0!)) \]

(A) Yes
(B) No
(C) I don’t know.
CQ 12 Is there a proof for implied (B)?

We used \((y = z!)\) as the invariant.

**CQ 12:** Is there a proof for implied (B)?

\[
(((y = z!) \land (z < x)) \rightarrow (y \ast (z + 1) = (z + 1)!))
\]

(A) Yes

(B) No

(C) I don't know.
We used \((y = z!)\) as the invariant.

**CQ 13:** Is there a proof for implied (C)?

\[(((((y = z!) \land \neg(z < x))) \rightarrow (y = x!))\]

(A) Yes
(B) No
(C) I don’t know.
CQ 14 Is there a proof for implied (C)?

We used \(((y = z!) \land (z \leq x))\) as the invariant.

CQ 14: Is there a proof for implied (C)?

\[ (((y = z!) \land (z \leq x)) \land \lnot(z < x)) \rightarrow (y = x!) \]

(A) Yes

(B) No

(C) I don’t know.
CQ 15: Which invariant leads to a valid proof?

(A) \((z \leq x)\)

(B) \((y = z!)\)

(C) \(((z \leq x) \land (y = z!))\)

(D) Two of (A), (B), and (C).

(E) All of (A), (B), and (C).
Example 2: Summary of invariants

Which invariant leads to a valid proof?

- \((z \leq x)\) does NOT lead to a valid proof.
- \((y = z!)\) does NOT lead to a valid proof.
- \(((z \leq x) \land (y = z!))\) does lead to a valid proof.
Proving Termination

Find an integer expression that
- is non-negative before the loop starts, at every iteration of the loop, and after the loop ends.
- decreases by at least 1 at every iteration of the loop.

This integer expression is called a **variant** (something that changes).

The loop must terminate because a non-negative integer can decrease by 1 a finite number of times.
Example 2: Finding a variant

Prove that the following program terminates.

\[\begin{align*}
y &= 1; \\
z &= 0; \\
\textbf{while} \ (z < x) \ {\{} \\
\quad &z = z + 1; \\
\quad &y = y \times z; \\
\{ &\}
\end{align*}\]

How do we find a variant? The loop guard \((z < x)\) helps.
Example 2: Proof of Termination

Consider the variant \((x - z)\).

Before the loop starts, \((x - z) \geq 0\) because the precondition is \((x \geq 0)\) and the second assignment mutates \(z\) to be 0.

During every iteration of the loop, \((x - z)\) decreases by 1 because \(x\) does not change and \(z\) increases by 1.

Thus, \(x - z\) will eventually reach 0.

When \(x - z = 0\), the loop guard \(z < x\) will terminate the loop.
Revisiting the learning goals

By the end of this lecture, you should be able to:
Partial correctness for while loops
  ▶ Determine whether a given formula is an invariant for a while loop.
  ▶ Find an invariant for a given while loop.
  ▶ Prove that a Hoare triple is satisfied under partial correctness for a program containing while loops.

Total correctness for while loops
  ▶ Determine whether a given formula is a variant for a while loop.
  ▶ Find a variant for a given while loop.
  ▶ Prove that a Hoare triple is satisfied under total correctness for a program containing while loops.