

Predicate Logic: Peano Arithmetic

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Lecture 20

Outline

The Learning Goals

Properties of Equality

Using Logic to Model Number Theory

Revisiting the Learning Goals

Learning goals

By the end of this lecture, you should be able to:

- ▶ Write a formal deduction proof using rules for equality.
- ▶ Write a formal deduction proof for properties of natural numbers using formal deduction rules and Peano axioms.

Formal Deduction Rules for Equality

$(\approx -)$ If $\Sigma \vdash A(t_1)$ and $\Sigma \vdash t_1 \approx t_2$ then
 $\Sigma \vdash A'(t_2)$,
where $A'(t_2)$ results from $A(t_1)$
by replacing some (not necessarily all) occurrences
of t_1 by t_2 .

$(\approx +)$ $\emptyset \vdash \underline{u \approx u}$.

Proving Properties of Equality

(Reflexivity) $\forall x (x \approx x)$

(Symmetry) $\forall x \forall y ((x = y) \rightarrow (y = x))$

(Transitivity) $\forall x \forall y \forall z ((x = y) \wedge (y = z) \rightarrow (x = z))$

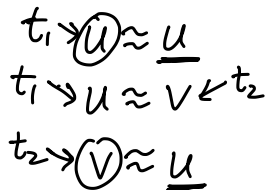
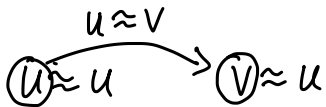
Proving Reflexivity

Reflexivity: $\emptyset \vdash \forall x(x \approx x)$

$$(1) \quad \emptyset \vdash u \approx u \quad (\approx +)$$

$$(2) \quad \emptyset \vdash \forall x (x \approx x) \quad (\forall +, 1)$$

Proving Symmetry



Symmetry: $\emptyset \vdash \forall x \forall y (x \approx y) \rightarrow (y \approx x)$

- (1) $\emptyset \vdash u \approx u$ ($\approx+$)
- (2) $u \approx v \vdash \underline{u \approx u}$ ($+$,
- (3) $u \approx v \vdash \underline{u \approx v}$ (ϵ)
- (4) $u \approx v \vdash \underline{v \approx u}$ ($\approx-, 2, 3, t_1 = u, t_2 = v, \quad$)
- (5) $\emptyset \vdash (u \approx v) \rightarrow (v \approx u)$ ($\rightarrow+$,
- (6) $\emptyset \vdash \forall y ((u \approx y) \rightarrow (y \approx u))$ ($\forall+$,
- (7) $\emptyset \vdash \forall x \forall y ((x \approx y) \rightarrow (y \approx x))$ ($\forall+$,

Proving Transitivity

$$\emptyset \vdash u \approx (u) (v) w$$

$$\dots \vdash u \approx v$$

$$\dots \vdash v \approx w$$

Transitivity: $\emptyset \vdash \forall x \forall y \forall z (x \approx y) \wedge (y \approx z) \rightarrow (x \approx z)$

$$1 (u \approx v) \wedge (v \approx w) \vdash (u \approx v) \wedge (v \approx w) \quad (E)$$

$$2 (u \approx v) \wedge (v \approx w) \vdash \underline{u \approx v} \quad (\wedge-, 1)$$

$$3 (u \approx v) \wedge (v \approx w) \vdash \underline{v \approx w} \quad (\wedge-, 1)$$

$$t_1 = v, t_2 = w, A(x) = u \approx x$$

$$4 (u \approx v) \wedge (v \approx w) \vdash \underline{u \approx w} \quad (\approx-, 2, 3)$$

$$5 \emptyset \vdash (u \approx v) \wedge (v \approx w) \rightarrow (u \approx w) \quad (\rightarrow+, \wedge, \forall+, \forall+, \forall+, \wedge)$$

$$6 \emptyset \vdash \forall x \forall y \forall z ((x \approx y) \wedge (y \approx z) \rightarrow (x \approx z)) \quad (\forall)$$

Outline

The Learning Goals

Properties of Equality

Using Logic to Model Number Theory

Revisiting the Learning Goals

Using Logic to Model Mathematics

We want to use predicate logic to model mathematics.

- ▶ Number theory, with 0 , $+$, and \cdot
- ▶ Set theory, with \in and \emptyset
- ▶ Group theory
- ▶ Graph theory
- ▶ Geometry

For each domain,

- ▶ Define axioms that describe the functions, predicates/relations and individuals/constants.
- ▶ Prove theorems in that domain using predicate logic.

Number Theory

We would like to formalize the properties of natural numbers.

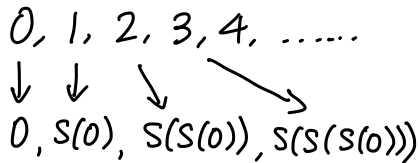
- ▶ The domain is the set of natural numbers, $0, 1, 2, 3, \dots$.
- ▶ Functions: addition $+$ and multiplication \cdot .
- ▶ Relations: ordering $<$.

The axioms should be a small set of true statements from which we can derive theorems about natural numbers.

Symbols for Number Theory

▶ Individual/Constant: 0

▶ Functions:
addition $+$
multiplication \cdot
successor $s(x)$



Peano Axioms (1/2)

Axioms for successor

PA1 Zero is not a successor of any natural number.

$\forall x (\neg (S(x) \approx 0))$ *S(x) is a successor of x.*

PA2 If two numbers are the same, *x is a predecessor of S(x).*
they must have the same predecessor.

$\forall x \forall y (S(x) \approx S(y) \rightarrow x \approx y)$

Axioms for addition

PA3 Adding zero to any number yields the same number.

$\forall x (x + 0 \approx x)$

PA4 Adding the successor of a number *to any natural number*
yields *the successor of adding the number to the same natural number.*

$\forall x \forall y (x + S(y) \approx S(x + y))$

Peano Axioms (2/2)

Axioms for multiplication

PA5 Multiplying a number by zero yields zero.

$$\forall x (x \cdot 0 \approx 0)$$

PA6 Multiplying one number and the successor of another number equals the product of the two numbers plus the first number.

$$\forall x \forall y (x \cdot s(y) \approx x \cdot y + x)$$

Axiom for Induction

For $n \in \mathbb{N}$, let $P(n)$ denote that n has the property P .

► (Base Case)

Prove that $P(0)$ is true.

► (Inductive Step) ← ~~Assume $P(k)$ is true for all $k \in \mathbb{N}$.~~

Assume that $P(k)$ is true for some $k \in \mathbb{N}$.

Prove that $P(k+1)$ is true. every

By the principle of mathematical induction, $P(n)$ is true for every $n \in \mathbb{N}$.

Expressing this in predicate logic:

$$(P(0) \wedge \forall x (P(x) \rightarrow P(s(x)))) \rightarrow \forall x P(x)$$

Axiom for Induction

$$\textcircled{1} \quad \phi \vdash A(0)$$

$$\textcircled{2} \quad \phi \vdash \forall x (A(x) \rightarrow A(s(x)))$$

$$\textcircled{3} \quad \phi \vdash A(0) \wedge \forall x (A(x) \rightarrow A(s(x))) \quad (\wedge+, 1, 2)$$

Axiom for induction

$$\textcircled{4} \quad \text{PA7 For each predicate formula } A(x) \text{ with free variable } x$$
$$\phi \vdash \underbrace{A(0)}_{\text{Base case}} \wedge \underbrace{\forall x (A(x) \rightarrow A(s(x)))}_{\text{Induction step}} \rightarrow \underbrace{\forall x A(x)}_{\text{Conclusion}} \quad (\text{PA 7}).$$

$$\textcircled{5} \quad \phi \vdash \forall x A(x) \quad (\rightarrow-, 3, 4)$$

Example 1: Every number is not equal to its successor

$$\text{PA2 } \forall x \forall y \quad S(x) \approx S(y) \rightarrow \underline{x} \approx \underline{y}$$

$$S(S(k)) \approx S(k) \rightarrow S(k) \approx k$$

Every natural number is not equal to its successor.

Prove that $\forall x (\neg(s(x) = x))$. $\phi \vdash (\forall x) (\neg(s(x) \approx 0))$ (PA1)

- ▶ Base Case: Prove that $\neg(s(0) = 0) \rightarrow 0 \quad \neg(s(0) \approx 0)$

Which Peano Axiom can we use to prove this?

- ▶ Induction Step:

Consider some $k \in \mathbb{N}$.

$$\neg(s(x) \approx x)$$

Assume that $\neg(s(k) = k)$.

$$\neg(s(s(k)) \approx s(k))$$

Prove that $\neg(s(s(k)) = s(k))$.

Which Peano Axiom can we use to prove this?

$$\neg(s(k) \approx k) \rightarrow \neg(s(s(k)) \approx s(k))$$

$$S(S(k)) \approx S(k) \rightarrow S(k) \approx k$$

Prove that $\forall x (\neg (S(x) \approx x))$ ←

- (1) $\phi \vdash \forall x (\neg (S(x) \approx 0))$ (PA 1)
 (2) $\phi \vdash \neg (S(0) \approx 0)$ ($\forall-$, 1)

$\phi \vdash$
 $\phi \vdash \forall x (\neg (S(x) \approx x) \rightarrow \neg (S(S(x)) \approx S(x)))$ ($\forall+$,

PA 7 $[A(0) \wedge \forall x (A(x) \rightarrow A(S(x)))] \rightarrow \forall x A(x)$

Prove that $\forall x (\neg (s(x) \approx x))$
 $F(x)$

Base case: Prove that $F(0)$ i.e. $\neg (s(0) \approx 0)$

Induction step: Prove that $\forall x (F(x) \rightarrow F(s(x)))$
induction hypothesis need to prove this.

$$\forall x (\neg (s(x) \approx x) \rightarrow \neg (s(s(x)) \approx s(x)))$$

Base case:

- (1) $\phi \vdash \forall x (\neg (s(x) \approx 0))$ (PA1)
(2) $\phi \vdash \neg (s(0) \approx 0)$ (\forall -, 1)

Induction step:

- (3) $\phi \vdash \forall x \forall y (s(x) \approx s(y) \rightarrow x \approx y)$ (PA2)
(4) $\phi \vdash \forall y (s(s(u)) \approx s(y) \rightarrow s(u) \approx y)$ (\forall -, 3)
(5) $\phi \vdash s(s(u)) \approx s(u) \rightarrow s(u) \approx u$ (\forall -, 4)
(6) $\neg (s(u) \approx u), s(s(u)) \approx s(u) \vdash s(s(u)) \approx s(u) \rightarrow s(u) \approx u$ (\rightarrow -, 5)
(7) $\neg (s(u) \approx u), s(s(u)) \approx s(u) \vdash s(s(u)) \approx s(u)$ (E)
(8) $\neg (s(u) \approx u), s(s(u)) \approx s(u) \vdash s(u) \approx u$ (\rightarrow -, 6, 7)
(9) $\neg (s(u) \approx u), s(s(u)) \approx s(u) \vdash \neg (s(u) \approx u)$ (E)
(10) $\neg (s(u) \approx u) \vdash \neg (s(s(u)) \approx s(u))$ (\neg +, 8, 9)
(11) $\phi \vdash \neg (s(u) \approx u) \rightarrow \neg (s(s(u)) \approx s(u))$ (\rightarrow +, 10)
(12) $\phi \vdash \forall x (\neg (s(x) \approx x) \rightarrow \neg (s(s(x)) \approx s(x)))$ (\forall +, 11)



$$(13) \quad \phi \vdash \left[\neg (s(0) \approx 0) \wedge \forall x (\neg (s(x) \approx x) \rightarrow \neg (s(s(x)) \approx s(x))) \right] \\ \rightarrow \forall x (\neg (s(x) \approx x)) \quad (\text{PA } 7)$$

$$(14) \quad \phi \vdash \neg (s(0) \approx 0) \wedge \forall x (\neg (s(x) \approx x) \rightarrow \neg (s(s(x)) \approx s(x))) \\ (\wedge +, 2, 12)$$

$$(15) \quad \phi \vdash \forall x (\neg (s(x) \approx x)) \quad (\rightarrow -, 13, 14).$$

Example 1: Every number is not equal to its successor

Every natural number is not equal to its successor.

Prove that $\forall x (\neg(s(x) = x))$.

1. $\emptyset \vdash \forall x (s(x) \neq 0)$ (PA1)
2. $\emptyset \vdash s(0) \neq 0$ (\forall -, 1)
3. $s(u) \neq u, s(s(u)) = s(u) \vdash s(s(u)) = s(u)$ (\in)
4. $\emptyset \vdash \forall x \forall y (s(x) = s(y) \rightarrow x = y)$ (PA2)
5. $\emptyset \vdash s(s(u)) = s(u) \rightarrow s(u) = u$ (\forall -, 4)
6. $s(u) \neq u, s(s(u)) = s(u) \vdash s(s(u)) = s(u) \rightarrow s(u) = u$ ($+$, 5)
7. $s(u) \neq u, s(s(u)) = s(u) \vdash s(u) = u$ (\rightarrow -, 6, 3)
8. $s(u) \neq u, s(s(u)) = s(u) \vdash s(u) \neq u$ (\in)
9. $s(u) \neq u \vdash s(s(u)) \neq s(u)$ (\neg +, 7, 8)
10. $\emptyset \vdash s(u) \neq u \rightarrow s(s(u)) \neq s(u)$ (\rightarrow +, 9)
11. $\emptyset \vdash \forall x (s(x) \neq x \rightarrow s(s(x)) \neq s(x))$ (10, \forall +, no u elsewhere)

Example 1: Every number is not equal to its successor

Every natural number is not equal to its successor.

Prove that $\forall x(\neg(s(x) = x))$. (continued)

$$12. \emptyset \vdash s(0) \neq 0 \wedge \forall x(s(x) \neq x \rightarrow s(s(x)) \neq s(x)) \quad (\wedge+, 2, 11)$$

$$13. \emptyset \vdash s(0) \neq 0 \wedge \forall x(s(x) \neq x \rightarrow s(s(x)) \neq s(x)) \\ \rightarrow \forall x(s(x) \neq x) \quad (PA7, \text{ with } A(x) : \text{" } s(x) \neq x \text{"})$$

$$14. \emptyset \vdash \forall x(s(x) \neq x) \quad (\rightarrow -, 12, 13)$$

Example 2: Every non-zero natural number has a predecessor

Every non-zero natural number has a predecessor.

Prove that $\forall x(x = 0 \vee \exists y (s(y) = x))$

Base case:

1. $\emptyset \vdash 0 = 0$ ($\approx +$)

2. $\emptyset \vdash 0 = 0 \vee \exists y(s(y) = 0)$ ($\vee +, 1$)

Induction step:

3. $k = 0 \vdash k = 0$ (\in)

4. $\emptyset \vdash s(k) = s(k)$ (prove separately using ($\approx +$))

5. $k = 0 \vdash s(k) = s(k)$ ($+, 4$)

6. $k = 0 \vdash s(0) = s(k)$ ($\approx -$) with $A(x) : \text{" } s(x) = s(k) \text{"}$

7. $k = 0 \vdash \exists y(s(y) = s(k))$ ($(\exists +), 6$)

8. $k = 0 \vdash (s(k) = 0) \vee \exists y(s(y) = s(k))$ ($\vee +, 8$)

Prove that $\forall x (x \approx 0 \vee \exists y (s(y) \approx x))$
 $F(x)$

Base case: Prove that $F(0)$
 $0 \approx 0 \vee \exists y (s(y) \approx 0)$

Induction step: Prove that $\forall x (F(x) \rightarrow F(s(x)))$

$\forall x (x \approx 0 \vee \exists y (s(y) \approx x) \rightarrow$
 $s(x) \approx 0 \vee \exists y (s(y) \approx s(x)))$

Base case (1) $\emptyset \vdash 0 \approx 0$ ($\approx +$)
 (2) $\emptyset \vdash 0 \approx 0 \vee \exists y (s(y) \approx 0)$ ($\vee +, 1$)

Induction step:

(16) $(k \approx 0) \vee \exists y (s(y) \approx k) \vdash s(k) \approx 0 \vee \exists y (s(y) \approx s(k))$ ($\vee -,$

(17) $\emptyset \vdash (k \approx 0 \vee \exists y (s(y) \approx k)) \rightarrow$
 $(s(k) \approx 0 \vee \exists y (s(y) \approx s(k)))$ ($\rightarrow +,$

(18) $\emptyset \vdash \forall x (x \approx 0 \vee \exists y (s(y) \approx x)) \rightarrow$
 $s(x) \approx 0 \vee \exists y (s(y) \approx s(x))$ ($\forall +,$

Induction step, first case of ($\vee -$):

(3) $\emptyset \vdash s(k) \approx s(k)$ ($\approx +$)

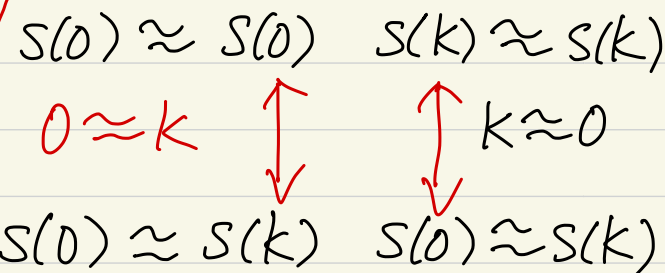
(4) $k \approx 0 \vdash s(k) \approx s(k)$ ($+ , 3$)

(5) $k \approx 0 \vdash k \approx 0$ (ϵ)

(6) $k \approx 0 \vdash s(0) \approx s(k)$ ($\approx -$, $t_1 = k, t_2 = 0, A(x) = s(x) \approx s(k)$)

(7) $k \approx 0 \vdash \exists y (s(y) \approx s(k))$ ($\exists +, 6$)

(8) $k \approx 0 \vdash s(k) \approx 0 \vee \exists y (s(y) \approx s(k))$ ($\vee +, 7$)



Example 2:

Inductive step, second case of $(\forall-)$

$$9. s(u) = k \vdash s(u) = k \text{ (}\in\text{)}$$

$$10. s(u) = k \vdash k = s(u) \text{ (9, symmetry of =)}$$

$$11. s(u) = k \vdash s(k) = s(k) \text{ (4, +)}$$

$$12. s(u) = k \vdash s(s(u)) = s(k) \text{ (11, 10, }(\approx -), A(x) : s(x) = s(k)\text{)}$$

$$13. s(u) = k \vdash \exists y(s(y) = s(k)) \text{ (12, } \exists+\text{)}$$

$$14. s(u) = k \vdash s(k) = 0 \vee \exists y(s(y) = s(k)) \text{ (13, } \vee+\text{)}$$

$$15. \exists y(s(y) = k) \vdash s(k) = 0 \vee \exists y(s(y) = s(k)) \text{ (14, } \exists-\text{),}$$

no u elsewhere)

$$16. k = 0 \vee \exists y(s(y) = k) \vdash s(k) = 0 \vee \exists y(s(y) = s(k)) \text{ (8, 15, } \vee-\text{)}$$

$$17. \emptyset \vdash k = 0 \vee \exists y(s(y) = k) \rightarrow s(k) = 0 \vee \exists y(s(y) = s(k)) \text{ (16, } \rightarrow+\text{)}$$

$$18. \emptyset \vdash \forall x(P(x) \rightarrow P(s(x))) \text{ (17, } \forall+\text{, no } k \text{ elsewhere)}$$

$$19. \emptyset \vdash P(0) \wedge \forall x(P(x) \rightarrow P(s(x))) \text{ (18, 2, } \wedge+\text{)}$$

$$20. \emptyset \vdash P(0) \wedge \forall x(P(x) \rightarrow P(s(x))) \rightarrow \forall xP(x) \text{ (PA7)}$$

$$21. \emptyset \vdash \forall x(x = 0 \vee \exists y(s(y) = x)) \text{ (20, 19, } \rightarrow-\text{)}$$

Revisiting the learning goals

By the end of this lecture, you should be able to:

- ▶ Write a formal deduction proof using rules for equality.
- ▶ Write a formal deduction proof for properties of natural numbers using formal deduction rules and Peano axioms.