Predicate Logic: Peano Arithmetic

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Lecture 20

Outline

Learning goals

By the end of this lecture, you should be able to:

- Write a formal deduction proof using rules for equality.
- Write a formal deduction proof for properties of natural numbers using formal deduction rules and Peano axioms.

Formal Deduction Rules for Equality

$(\approx -)$ If $\Sigma \vdash \boxed{A(t_1)}$ and $\Sigma \vdash \boxed{t_1 \approx t_2}$ then
$\Sigma \vdash \boxed{A'(t_2)}$,
where $A'(t_2)$ results from $A(t_1)$
by replacing some (not necessarily all) occurrences
of t_1 by t_2.

$(\approx +)$ $\emptyset \vdash u \approx u$.

Proving Properties of Equality

(Reflexivity) $\forall x \ (x = x)$
(Symmetry) $\forall x \ \forall y \ ((x = y) \to (y = x))$
(Transitivity) $\forall x \ \forall y \ \forall z \ ((x = y) \land (y = z) \to (x = z))$

Proving Reflexivity

Reflexivity: $\emptyset \vdash \forall x(x \approx x)$

(1) $\emptyset \vdash u \approx u$ $(\approx +)$

(2) $\emptyset \vdash \forall x \, (x \approx x)$ $(\forall +, 1)$

Proving Symmetry

$$A(t_1) \to \underline{u \approx u} \quad (\approx + ☆$$

$t_1 \approx t_2$

$u \approx v$

$$u \approx v \; (\in) ☆$$

$$A(t_2) \to \underline{v \approx u} \quad (\approx -,$$

Symmetry: $\emptyset \vdash \forall x \, \forall y \, (x \approx y) \to (y \approx x)$

(1) $\emptyset \vdash u \approx u \quad (\approx +)$

(2) $u \approx v \vdash u \approx u \quad (+, 1)$

(3) $u \approx v \vdash u \approx v \quad (\in)$

$A(x) = x \approx u$

(4) $u \approx v \vdash v \approx u \quad (\approx -, 2, 3, t_1 = u, t_2 = v, \quad)$

(5) $\emptyset \vdash (u \approx v) \to (v \approx u) \quad (\to +, 4)$

(6) $\emptyset \vdash \forall y \, (u \approx y) \to (y \approx u) \quad (\forall +, 5)$

(7) $\emptyset \vdash \forall x \, \forall y \, (x \approx y) \to (y \approx x) \quad (\forall +, 6)$

Proving Transitivity

$u \approx u$
$u \approx v$
$u \approx v$
$v \approx w$
$u \approx w$

$A(t_1) \to u \approx v \Leftarrow t_1$

$A(x) = u \approx x \quad v \approx w \quad t_1 \approx t_2$

$A(t_2) \to u \approx w \Leftarrow t_2$

$v \approx w$
$v \approx u$
$u \approx v$
$u \approx w$

Transitivity: $\emptyset \vdash \forall x\ \forall y\ \forall z\ (x \approx y) \wedge (y \approx z) \to (x \approx z)$

1 $(u \approx v) \wedge (v \approx w) \vdash (u \approx v) \wedge (v \approx w)$ $(\in)$

2 $(u \approx v) \wedge (v \approx w) \vdash u \approx v$ $(\wedge-, 1)$

3 $(u \approx v) \wedge (v \approx w) \vdash v \approx w$ $(\wedge-, 1)$

4 $(u \approx v) \wedge (v \approx w) \vdash (u \approx w)$ $(\approx-, 2, 3, t_1 = v, t_2 = w, A(x) = u \approx x)$

5 $\emptyset \vdash (u \approx v) \wedge (v \approx w) \to (u \approx w)$ $(\to+, 4)$

6 $\emptyset \vdash \forall x\ \forall y\ \forall z\ (x \approx y) \wedge (y \approx z) \to (x \approx z)$ $(\forall+, \forall+, \forall+, 5)$

Outline

The Learning Goals

Properties of Equality

Using Logic to Model Number Theory

Revisiting the Learning Goals

Using Logic to Model Mathematics

We want to use predicate logic to model mathematics.

- Number theory, with 0, $+$, and $\cdot$
- Set theory, with $\in$ and $\emptyset$
- Group theory
- Graph theory
- Geometry

For each domain,

- Define axioms that describe the functions, predicates/relations and individuals/constants.
- Prove theorems in that domain using predicate logic.

Number Theory

We would like to formalize the properties of natural numbers.

- The domain is the set of natural numbers, $0, 1, 2, 3, \ldots$.
- Functions: addition $+$ and multiplication $\cdot$.
- Relations: ordering $<$.

The axioms should be a small set of true statements from which we can derive theorems about natural numbers.

Symbols for Number Theory

$0, 1, 2, 3, 4, \ldots\ldots$

$0, S(0), S(S(0)), S(S(S(0))).$

- Individual/Constant: 0
- Functions:
  addition $+$
  multiplication $\cdot$
  successor $s(x)$

Peano Axioms (1/2)

Axioms for successor

PA1 Zero is not a successor of any natural number.

$$\forall x \neg (S(x) \approx 0)$$

PA2 If two numbers are the same, they must have the same predecessor.

If the successors of two numbers are the same, then the two numbers are the same.

$$\forall x \forall y (S(x) \approx S(y) \to x \approx y)$$

Axioms for addition

PA3 Adding zero to any number yields the same number.

$$\forall x (x + 0 \approx x)$$

PA4 Adding the successor of a number yields the successor of adding the number.

$$\forall x \forall y (x + S(y) \approx S(x + y))$$

Peano Axioms (2/2)

Axioms for multiplication

PA5 Multiplying a number by zero yields zero.

$$\forall x \; (x \cdot 0 \approx 0)$$

PA6 Multiplying one number and the successor of another number equals the product of the two numbers plus the first number.

$$\forall x \; \forall y \; (x \cdot s(y) \approx x \cdot y + x)$$

Axiom for Induction

For $n \in \mathbb{N}$, let $P(n)$ denote that n has the property P.

- (Base Case)
  Prove that $P(0)$ is true.
- (Inductive Step)
  Assume that $P(k)$ is true for some $k \in \mathbb{N}$.
  Prove that $P(\underbrace{k+1}_{S(k)})$ is true.

By the principle of mathematical induction, $P(n)$ is true for every $n \in \mathbb{N}$.

Expressing this in predicate logic:

$$(\underbrace{P(0)}_{\text{Base case}} \wedge \underbrace{\forall x\ (P(\overset{k}{x}) \to P(s(x)))}_{\text{Induction step}}) \to \underbrace{\forall x\ P(x)}_{\text{Conclusion}}$$

Axiom for Induction

(1) $\emptyset \vdash A(0)$ ......

(2) $\emptyset \vdash \forall x (A(x) \to A(s(x)))$ ......

(3) $\emptyset \vdash A(0) \land \forall x (A(x) \to A(s(x)))$ $(\land+, 1, 2)$

Axiom for induction

PA7 For each predicate formula $A(x)$ with free variable x

(4) $\emptyset \vdash [A(0) \land \forall x(A(x) \to A(s(x)))] \to \forall x A(x)$ (PA 7)

(5) $\emptyset \vdash \forall x A(x)$ $(\to-, 3, 4)$

Prove that $\forall x\ \underline{(\neg(S(x) \approx x))}$

PA7 $[A(0) \wedge \forall x\,(A(x) \to A(S(x)))] \to \forall x\ \underline{A(x)}$

$\phi \vdash \neg(S(0) \approx 0)$

$\phi \vdash \forall x\,(\neg(S(x) \approx x) \to \neg(S(S(x)) \approx S(x)))$

$\phi \vdash \neg(S(0) \approx 0) \wedge \forall x\,(\neg(S(x) \approx x) \to \neg(S(S(x)) \approx S(x)))$ ($\wedge +$,

$$\phi \vdash [\neg(S(0) \approx 0) \wedge \forall x\,(\neg(S(x) \approx x) \to \neg(S(S(x)) \approx S(x)))] \to \forall x\,(\neg(S(x) \approx x))$$ (PA7)

$\phi \vdash \forall x\,(\neg(S(x) \approx x))$ ($\to -$,

Example 1: Every number is not equal to its successor

Every natural number is not equal to its successor.
Prove that $\forall x\ (\neg(s(x) = x))$.

PA1 $\cancel{\forall x}(\neg(s(x) \approx 0))$

$\Updownarrow$

0

- Base Case: Prove that $\neg(s(0) = 0)$ 1

  Which Peano Axiom can we use to prove this?

- Induction Step:
  Consider some $k \in \mathbb{N}$.
  Assume that $\neg(s(k) = k)$.
  Prove that $\neg(s(s(k)) = s(k))$.

  Which Peano Axiom can we use to prove this?

$\neg(s(k) \approx k) \to \neg(s(s(k)) \approx s(k))$ 2

$s(s(k)) \approx s(k) \to s(k) \approx k$

$\forall x\, \forall y\quad s(x) \approx s(y) \to x \approx y$

(1) $\phi \vdash \forall x(\neg(S(x) \approx 0))$ PA1

(2) $\phi \vdash \neg(S(0) \approx 0)$ $(\forall-, 1)$

Example 1: Every number is not equal to its successor

Every natural number is not equal to its successor.
Prove that $\forall x\ (\neg(s(x) = x))$.

$1. \emptyset \vdash\ \forall x\ (s(x) \neq 0)\ (PA1)$

$2. \emptyset \vdash\ s(0) \neq 0\ (\forall-, 1)$

$3. s(u) \neq u, s(s(u)) = s(u) \vdash\ s(s(u)) = s(u)\ (\in)$

$4. \emptyset \vdash\ \forall x\ \forall y\ (s(x) = s(y) \rightarrow x = y)\ (PA2)$

$5. \emptyset \vdash\ s(s(u)) = s(u) \rightarrow s(u) = u\ (\forall-, 4)$

$6. s(u) \neq u, s(s(u)) = s(u) \vdash\ s(s(u)) = s(u) \rightarrow s(u) = u\ (+, 5)$

$7. s(u) \neq u, s(s(u)) = s(u) \vdash\ s(u) = u\ (\rightarrow -, 6, 3)$

$8. s(u) \neq u, s(s(u)) = s(u) \vdash\ s(u) \neq u\ (\in)$

$9. s(u) \neq u \vdash\ s(s(u)) \neq s(u)\ (\neg+, 7, 8)$

$10. \emptyset \vdash\ s(u) \neq u \rightarrow s(s(u)) \neq s(u)\ (\rightarrow +, 9)$

$11. \emptyset \vdash\ \forall x(s(x) \neq x \rightarrow s(s(x)) \neq s(x))$ $(10, \forall+,$ no u elsewhere$)$

Example 1: Every number is not equal to its successor

Every natural number is not equal to its successor.
Prove that $\forall x(\neg(s(x) = x))$. (continued)

$12. \emptyset \vdash\ s(0) \neq 0 \wedge \forall x(s(x) \neq x \to s(s(x)) \neq s(x))\ (\wedge+, 2, 11)$

$13. \emptyset \vdash\ s(0) \neq 0 \wedge \forall x(s(x) \neq x \to s(s(x)) \neq s(x))$
$\quad \to \forall x(s(x) \neq x)\ (PA7,$ with $A(x) :\text{``}\ s(x) \neq x'')$

$14. \emptyset \vdash\ \forall x(s(x) \neq x)\ (\to -, 12, 13)$

Example 2: Every non-zero natural number has a predecessor

Every non-zero natural number has a predecessor.
Prove that $\forall x(x = 0 \vee \exists y\ (s(y) = x))$

Base case:
$1.\emptyset \vdash\ 0 = 0\ (\approx +)$
$2.\emptyset \vdash\ 0 = 0 \vee \exists y(s(y) = 0)\ (\vee+, 1)$
Induction step:
$3.k = 0 \vdash\ k = 0\ (\in)$
$4.\emptyset \vdash\ s(k) = s(k)$ (prove separately using $(\approx +)$)
$5.k = 0 \vdash\ s(k) = s(k)\ (+, 4)$
$6.k = 0 \vdash\ s(0) = s(k)\ (\approx -)$ with $A(x) :``\ s(x) = s(k)''$
$7.k = 0 \vdash\ \exists y(s(y) = s(k))\ ((\exists+), 6)$
$8.k = 0 \vdash\ (s(k) = 0) \vee \exists y(s(y) = s(k))\ (\vee+, 8)$

Example 2:

Inductive step, second case of $(\vee-)$

$9. s(u) = k \vdash\ s(u) = k\ (\in)$

$10. s(u) = k \vdash\ k = s(u)$ $(9,$ symmetry of $=)$

$11. s(u) = k \vdash\ s(k) = s(k)\ (4, +)$

$12. s(u) = k \vdash\ s(s(u)) = s(k)\ (11, 10, (\approx -), A(x) : s(x) = s(k))$

$13. s(u) = k \vdash\ \exists y(s(y) = s(k))\ (12, \exists +)$

$14. s(u) = k \vdash\ s(k) = 0 \vee \exists y(s(y) = s(k))\ (13, \vee +)$

$15. \exists y(s(y) = k) \vdash\ s(k) = 0 \vee \exists y(s(y) = s(k))\ (14, \exists -),$
no u elsewhere)

$16. k = 0 \vee \exists y(s(y) = k) \vdash\ s(k) = 0 \vee \exists y(s(y) = s(k))\ (8, 15, \vee -)$

$17. \emptyset \vdash k = 0 \vee \exists y(s(y) = k) \to s(k) = 0 \vee \exists y(s(y) = s(k))(16, \to +)$

$18. \emptyset \vdash\ \forall x(P(x) \to P(s(x)))\ (17, \forall +,$ no k elsewhere)

$19. \emptyset \vdash\ P(0) \wedge \forall x(P(x) \to P(s(x)))\ (18, 2, \wedge +)$

$20. \emptyset \vdash\ P(0) \wedge \forall x(P(x) \to P(s(x))) \to \forall x P(x)\ (PA7)$

$21. \emptyset \vdash\ \forall x(x = 0 \vee \exists y(s(y) = x))\ (20, 19, \to -)$

Revisiting the learning goals

By the end of this lecture, you should be able to:

- Write a formal deduction proof using rules for equality.
- Write a formal deduction proof for properties of natural numbers using formal deduction rules and Peano axioms.