Predicate Logic: Formal Deduction

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Lecture 16
Outline

The Learning Goals

Forall-elimination

Exists-introduction

Forall-introduction

Exists-elimination

Putting them together

Revisiting the Learning Goals
Learning goals

By the end of this lecture, you should be able to:

▶ Describe the rules of inference for formal deduction for predicate logic.
▶ Prove that a conclusion follows from a set of premises using formal deduction inference rules.
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Forall-elimination

\[ \forall\text{-elimination (}\forall\neg\text{)} \]

if \( \Sigma \vdash \forall x A(x) \),
then \( \Sigma \vdash A(t) \).

Compare this to \( \land\text{-elimination (}\land\neg\text{)} \)

if \( \Sigma \vdash A \land B \),
then \( \Sigma \vdash A \).

if \( \Sigma \vdash A \land B \),
then \( \Sigma \vdash B \).
Exercise: Forall-elimination

Show that

\[ P(u), \forall x (P(x) \rightarrow \neg Q(x)) \vdash \neg Q(u). \]
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∃-introduction (∃+)

if \( \Sigma \vdash A(t) \),
then \( \Sigma \vdash \exists x \ A(x) \).

where \( A(x) \) results by replacing some (not necessarily all) occurrences of \( t \) in \( A(t) \) by \( x \).

Compare this to ∨-introduction (∨+)

if \( \Sigma \vdash A \),
then \( \Sigma \vdash A \lor B \).

if \( \Sigma \vdash B \),
then \( \Sigma \vdash A \lor B \).
CQ Exists-introduction

Proof 1:

(1) \( \Sigma \vdash (P(v) \rightarrow Q(v)) \) by assumption
(2) \( \Sigma \vdash (\exists x (P(x) \rightarrow Q(v))) \) by \((\exists+, 4)\)

Proof 2:

(1) \( \Sigma \vdash (P(v) \rightarrow Q(v)) \) by assumption
(2) \( \Sigma \vdash (\exists x (P(x) \rightarrow Q(x))) \) by \((\exists+, 4)\)

Which of the following is a correct application of the \( \exists^+ \) rule?

(A) Both proofs
(B) Proof 1 only
(C) Proof 2 only
(D) Neither proof
Exercise: Exists-introduction

Show that

\[\{\neg P(v)\} \vdash (\exists x \ (P(x) \rightarrow Q(v))).\]
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For all-introduction

∀-introduction (\(\forall^+\))

if \(\Sigma \vdash A(u), \ u\) not occurring in \(\Sigma\),
then \(\Sigma \vdash \forall x A(x)\).

Compare this to ∧-introduction (\(\land^+\))

if \(\Sigma \vdash A\),
\(\Sigma \vdash B\),
then \(\Sigma \vdash A \land B\).
Exercise: Forall-introduction

Show that

\[(\forall x (P(x) \to Q(x))) \vdash ((\forall x P(x)) \to (\forall y Q(y))).\]
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Exists-elimination

∃-elimination ($\exists -$)

if $\Sigma, A(u) \vdash B$, $u$ not occurring in $\Sigma$ or $B$,
then $\Sigma, \exists x \ A(x) \vdash B$.

Compare this to $\lor$-elimination ($\lor -$)

if $\Sigma, A \vdash C$,
$\Sigma, B \vdash C$,
then $\Sigma, A \lor B \vdash C$. 
Exercise: Exists-elimination

Show that

$$\exists x \ (P(x) \lor Q(x)) \vdash (\exists x \ P(x) \lor (\exists x \ Q(x))).$$
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Show that

$$\exists y \ (\forall x \ P(x, y)) \vdash \forall x \ (\exists y \ P(x, y)).$$

Which rule should we apply next?

(A) $\forall+$
(B) $\forall−$
(C) $\exists+$
(D) $\exists−$
(E) Another rule
By the end of this lecture, you should be able to:

- Describe the rules of inference for formal deduction for predicate logic.
- Prove that a conclusion follows from a set of premises using formal deduction inference rules.