Predicate Logic: Logical Consequence

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Lecture 15

Outline

Learning goals

By the end of this lecture, you should be able to:

- Define logical consequence for predicate logic.
- Prove that a logical consequence holds.
- Prove that a logical consequence does not hold.

Definition of Logical Consequence

Define the symbols.

- Σ is a set of predicate formulas.
- A is a predicate formula.

$\Sigma \models A$
(Σ logically implies A)
(A is a logical consequence of Σ)
iff for every valuation v, if $\Sigma^v = 1$, then $A^v = 1$.

Prove a logical consequence

Consider the logical consequence $\Sigma \vDash A$.
To prove that the logical consequence holds, we need to consider

(A) Every valuation v such that $\Sigma^v = 1$.

(B) Every valuation v such that $\Sigma^v = 0$.

(C) One valuation v such that $\Sigma^v = 1$.

(D) One valuation v such that $\Sigma^v = 0$.

Disprove a logical consequence

Consider the logical consequence $\Sigma \models A$.
To prove that the logical consequence does NOT hold, we need to consider

(A) Every valuation v such that $\Sigma^v = 1$ and $A^v = 1$.

(B) Every valuation v such that $\Sigma^v = 1$ and $A^v = 0$.

(C) One valuation v such that $\Sigma^v = 1$ and $A^v = 1$.

(D) One valuation v such that $\Sigma^v = 1$ and $A^v = 0$.

Example: Prove/Disprove a logical consequence

Consider the logical consequence below.

$$\forall x \neg A(x) \vDash \neg(\exists x\, A(x)).$$

If the logical consequence holds, prove it.
If it does not hold, provide a counterexample.
Answer: This logical consequence holds.
Proof: We prove this by contradiction. Assume that there exists a valuation such that $(\forall x \neg A(x))^v = 1$ and $(\neg \exists x A(x))^v = 0$. Form $A(u)$ from $A(x)$, u not occurring in $A(x)$.
By $(\forall x \neg A(x))^v = 1$, we obtain $(\neg A(u))^{v(u/\alpha)} = 1$ for every $\alpha \in D$. Therefore, $A(u)^{v(u/\alpha)} = 0$ for every $\alpha \in D$. (1)
By $(\neg \exists x A(x))^v = 0$, we obtain $(\exists x A(x))^v = 1$. Thus, there exists $\beta \in D$ such that $A(u)^{v(u/\beta)} = 1$, which contradicts (1).
Hence, $\forall x \neg A(x) \vDash \neg \exists x A(x)$. QED

Example: Prove/Disprove a logical consequence

Consider the logical consequence below.

$$\forall x(A(x) \to B(x)) \vDash \forall x A(x) \to \forall x B(x)$$

If the logical consequence holds, prove it.
If it does not hold, provide a counterexample.
Answer: This logical consequence holds.
Proof: We prove this by contradiction. Assume that there exists a valuation such that $(\forall x(A(x) \to B(x)))^v = 1$ and $(\forall x A(x) \to \forall x B(x))^v = 0$. By $(\forall x A(x) \to \forall x B(x))^v = 0$, $(\forall x A(x))^v = 1$ and $(\forall x B(x))^v = 0$. Form $A(u)$ and $B(u)$, u not occurring in $A(x)$ or in $B(x)$. By $(\forall x A(x))^v = 1$, $A(u)^{v(u/\alpha)} = 1$ for every $\alpha \in D$. (1) By $(\forall x B(x))^v = 0$, $B(u)^{v(u/\alpha)} = 0$ for every $\alpha \in D$. (2) By $(\forall x(A(x) \to B(x)))^v = 1$, $(A(u) \to B(u))^{v(u/\alpha)} = 1$ for every $\alpha \in D$. (3) By (1) and (3), we have that $B(u)^{v(u/\alpha)} = 1$ for every $\alpha \in D$, which contradicts (2). Hence, the logical consequence holds. QED

Example: Prove/Disprove a logical consequence

Consider the logical consequence below.

$$\forall x A(x) \to \forall x B(x) \vDash \forall x(A(x) \to B(x))$$

If the logical consequence holds, prove it.
If it does not hold, provide a counterexample.
Answer: This logical consequence does not hold.
Consider the valuation v: $D = \{1, 2\}$. $A^v = \{1\}$. $B^v = \emptyset$. Form $A(u)$ and $B(u)$ for u not occurring in $A(x)$ and $B(x)$.
Since $2 \notin A^v$, $A(u)^{v(u/2)} = 0$. Thus, $(\forall x A(x))^v = 0$ and $(\forall x A(x) \to \forall x B(x))^v = 1$.
Since $1 \in A^v$ and $1 \notin B^v$, $A(u)^{v(u/1)} = 1$ and $B(u)^{v(u/1)} = 0$.
Thus, $(A(u) \to B(u))^{v(u/1)} = 0$. Therefore, $(\forall x(A(x) \to B(x)))^v = 0$.
Thus, the logical consequence does not hold. QED

Example: Prove/Disprove the logical consequence

Prove the following

$$\exists x(A(x) \wedge B(x)) \vDash \exists x A(x) \wedge \exists x B(x).$$

We prove this by contradiction. Assume that there exists a valuation v such that $(\exists x(A(x) \wedge B(x)))^v = 1$ and $(\exists x A(x) \wedge \exists x B(x))^v = 0$.
Form $A(u)$ and $B(u)$ for u not occurring in $A(x)$ and $B(x)$.
By $(\exists x(A(x) \wedge B(x)))^v = 1$, $(A(u) \wedge B(u))^{v(u/\alpha)} = 1$ for a particular $\alpha \in D$. (1)
By $(\exists x A(x) \wedge \exists x B(x))^v = 0$, $(\exists x A(x))^v = 0$ and $(\exists x B(x))^v = 0$.
By $(\exists x A(x))^v = 0$, $A(u)^{v(u/\alpha)} = 0$ for every $\alpha \in D$. By $(\exists x B(x))^v = 0$, $B(u)^{v(u/\alpha)} = 0$ for every $\alpha \in D$. Therefore, for every $\alpha \in D$, $(A(u) \wedge B(u))^{v(u/\alpha)} = 0$, which contradicts (1).
QED

Example: Prove/Disprove the logical consequence

Prove the following

$$\exists x A(x) \wedge \exists x B(x) \not\models \exists x (A(x) \wedge B(x)).$$

Consider the valuation v: $D = \{1, 2\}$. $A^v = \{1\}$. $B^v = \{2\}$. Form $A(u)$ and $B(u)$ for u not occurring in $A(x)$ and $B(x)$.

Since $1 \in A^v$, $A(u)^{v(u/1)} = 1$. Thus, $(\exists x A(x))^v = 1$. Since $2 \in B^v$, $B(u)^{v(u/2)} = 1$. Thus, $(\exists x B(x))^v = 1$. Therefore, $(\exists x A(x) \wedge \exists x B(x))^v = 1$.

Since $1 \notin B^v$, $(A(u) \wedge B(u))^{v(u/1)} = 0$. Since $2 \notin A^v$, $(A(u) \wedge B(u))^{v(u/2)} = 0$. Therefore, $(\exists x (A(x) \wedge B(x)))^v = 0$.

Revisiting the learning goals

By the end of this lecture, you should be able to:

- Define logical consequence for predicate logic.
- Prove that a logical consequence holds.
- Prove that a logical consequence does not hold.