Predicate Logic:
Logical Consequence

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Lecture 15
Outline

The Learning Goals

Definition of Logical Consequence

Proving/Disproving a Logical Consequence

Revisiting the Learning Goals
Learning goals

By the end of this lecture, you should be able to:

▶ Define logical consequence for predicate logic.
▶ Prove that a logical consequence holds.
▶ Prove that a logical consequence does not hold.
Definition of Logical Consequence

Define the symbols.

- $\Sigma$ is a set of predicate formulas.
- $A$ is a predicate formula.

$\Sigma \models A$

($\Sigma$ logically implies $A$)

($A$ is a logical consequence of $\Sigma$)

iff for every valuation $v$, if $\Sigma^v = 1$, then $A^v = 1$. 
Prove a logical consequence

Consider the logical consequence $\Sigma \vdash A$.
To prove that the logical consequence holds, we need to consider

(A) Every valuation $v$ such that $\Sigma^v = 1$.
(B) Every valuation $v$ such that $\Sigma^v = 0$.
(C) One valuation $v$ such that $\Sigma^v = 1$.
(D) One valuation $v$ such that $\Sigma^v = 0$. 
Consider the logical consequence $\Sigma \models A$.
To prove that the logical consequence does NOT hold, we need to consider

(A) Every valuation $v$ such that $\Sigma^v = 1$ and $A^v = 1$.
(B) Every valuation $v$ such that $\Sigma^v = 1$ and $A^v = 0$.
(C) One valuation $v$ such that $\Sigma^v = 1$ and $A^v = 1$.
(D) One valuation $v$ such that $\Sigma^v = 1$ and $A^v = 0$. 
Example: Prove/Disprove a logical consequence

Consider the logical consequence below.

\[ \forall x \neg A(x) \models \neg (\exists x A(x)) . \]

If the logical consequence holds, prove it.
If it does not hold, provide a counterexample.

Answer: This logical consequence holds.

Proof: We prove this by contradiction. Assume that there exists a valuation such that \((\forall x \neg A(x))^v = 1\) and \((\neg \exists x A(x))^v = 0\). Form \(A(u)\) from \(A(x)\), \(u\) not occurring in \(A(x)\).

By \((\forall x \neg A(x))^v = 1\), we obtain \((\neg A(u))^{v(u/\alpha)} = 1\) for every \(\alpha \in D\). Therefore, \(A(u)^{v(u/\alpha)} = 0\) for every \(\alpha \in D\). (1)

By \((\neg \exists x A(x))^v = 0\), we obtain \((\exists x A(x))^v = 1\). Thus, there exists \(\beta \in D\) such that \(A(u)^{v(u/\beta)} = 1\), which contradicts (1).

Hence, \(\forall x \neg A(x) \models \neg \exists x A(x)\). QED
Example: Prove/Disprove a logical consequence

Consider the logical consequence below.

$$\forall x(A(x) \rightarrow B(x)) \models \forall x A(x) \rightarrow \forall x B(x)$$

If the logical consequence holds, prove it.
If it does not hold, provide a counterexample.

Answer: This logical consequence holds.

Proof: We prove this by contradiction. Assume that there exists a valuation such that
$$(\forall x(A(x) \rightarrow B(x)))^v = 1$$
and
$$(\forall x A(x) \rightarrow \forall x B(x))^v = 0.$$ By $$(\forall x A(x) \rightarrow \forall x B(x))^v = 0,$$
$$(\forall x A(x))^v = 1$$ and $$(\forall x B(x))^v = 0.$$ Form $$A(u)$$ and $$B(u), u$$ not occurring in $$A(x)$$ or in $$B(x).$$ By $$(\forall x A(x))^v = 1, A(u)^v(u/\alpha) = 1$$ for every $$\alpha \in D.$$ (1) By $$(\forall x B(x))^v = 0, B(u)^v(u/\alpha) = 0$$ for every $$\alpha \in D.$$ (2) By $$(\forall x(A(x) \rightarrow B(x)))^v = 1,$$
$$(A(u) \rightarrow B(u))^v(u/\alpha) = 1$$ for every $$\alpha \in D.$$ (3) By (1) and (3), we have that $$B(u)^v(u/\alpha) = 1$$ for every $$\alpha \in D,$$ which contradicts (2). Hence, the logical consequence holds. QED
Consider the logical consequence below.

$$\forall x A(x) \rightarrow \forall x B(x) \models \forall x (A(x) \rightarrow B(x))$$

If the logical consequence holds, prove it.
If it does not hold, provide a counterexample.
Answer: This logical consequence does not hold.
Consider the valuation $v$: $D = \{1, 2\}$. $A^v = \{1\}$. $B^v = \emptyset$. Form $A(u)$ and $B(u)$ for $u$ not occurring in $A(x)$ and $B(x)$.
Since $2 \notin A^v$, $A(u)^v(u/2) = 0$. Thus, $(\forall x A(x))^v = 0$ and $(\forall x A(x) \rightarrow \forall x B(x))^v = 1$.
Since $1 \in A^v$ and $1 \notin B^v$, $A(u)^v(u/1) = 1$ and $B(u)^v(u/1) = 0$. Thus, $(A(u) \rightarrow B(u))^v(u/1) = 0$. Therefore, $(\forall x (A(x) \rightarrow B(x)))^v = 0$.
Thus, the logical consequence does not hold. QED
Example: Prove/Disprove the logical consequence

Prove the following

\[ \exists x (A(x) \land B(x)) \models \exists x A(x) \land \exists x B(x). \]

We prove this by contradiction. Assume that there exists a valuation \( v \) such that \((\exists x (A(x) \land B(x)))^v = 1\) and 
\((\exists x A(x) \land \exists x B(x))^v = 0\).

Form \( A(u) \) and \( B(u) \) for \( u \) not occurring in \( A(x) \) and \( B(x) \).

By \((\exists x (A(x) \land B(x)))^v = 1\), \((A(u) \land B(u))^v(u/\alpha) = 1\) for a particular \( \alpha \in D \). (1)

By \((\exists x A(x) \land \exists x B(x))^v = 0\), \((\exists x A(x))^v = 0\) and \((\exists x B(x))^v = 0\).

By \((\exists x A(x))^v = 0\), \( A(u)^v(u/\alpha) = 0\) for every \( \alpha \in D \). By
\((\exists x B(x))^v = 0\), \( B(u)^v(u/\alpha) = 0\) for every \( \alpha \in D \). Therefore, for every \( \alpha \in D \), 
\((A(u) \land B(u))^v(u/\alpha) = 0\), which contradicts (1).

QED
Example: Prove/Disprove the logical consequence

Prove the following

\[ \exists x A(x) \land \exists x B(x) \not\equiv \exists x (A(x) \land B(x)). \]

Consider the valuation \( v: D = \{1, 2\} \). \( A^v = \{1\} \). \( B^v = \{2\} \). Form \( A(u) \) and \( B(u) \) for \( u \) not occurring in \( A(x) \) and \( B(x) \).

Since \( 1 \in A^v \), \( A(u)^v(u/1) = 1 \). Thus, \( (\exists x A(x))^v = 1 \). Since 2 \( \in B^v \), \( B(u)^v(u/2) = 1 \). Thus, \( (\exists x B(x))^v = 1 \). Therefore, \( (\exists x A(x) \land \exists x B(x))^v = 1 \).

Since \( 1 \notin B^v \), \( (A(u) \land B(u))^v(u/1) = 0 \). Since 2 \( \notin A^v \),  
\[ (A(u) \land B(u))^v(u/2) = 0. \]
Therefore, \( (\exists x (A(x) \land B(x))^v = 0. \)
Revisiting the learning goals

By the end of this lecture, you should be able to:

- Define logical consequence for predicate logic.
- Prove that a logical consequence holds.
- Prove that a logical consequence does not hold.