Predicate Logic: Logical Consequence

Alice Gao

Lecture 15
Outline

The Learning Goals

Definition of Logical Consequence

Proving/Disproving a Logical Consequence

Revisiting the Learning Goals
Learning goals

By the end of this lecture, you should be able to:

▶ Define logical consequence for predicate logic.
▶ Prove that a logical consequence holds.
▶ Prove that a logical consequence does not hold.
Definition of Logical Consequence

Define the symbols.

- $\Sigma$ is a set of predicate formulas.
- $A$ is a predicate formula.

$\Sigma \models A$

($\Sigma$ logically implies $A$)

($A$ is a logical consequence of $\Sigma$)

iff for every valuation $v$, if $\Sigma^v = 1$, then $A^v = 1$. 
Prove a logical consequence

Consider the logical consequence \( \Sigma \vdash A \).
To prove that the logical consequence holds, we need to consider

(A) Every valuation \( v \) such that \( \Sigma^v = 1 \).
(B) Every valuation \( v \) such that \( \Sigma^v = 0 \).
(C) One valuation \( v \) such that \( \Sigma^v = 1 \).
(D) One valuation \( v \) such that \( \Sigma^v = 0 \).
Consider the logical consequence \( \Sigma \models A \).
To prove that the logical consequence does NOT hold, we need to consider

(A) Every valuation \( v \) such that \( \Sigma^v = 1 \) and \( A^v = 1 \).
(B) Every valuation \( v \) such that \( \Sigma^v = 1 \) and \( A^v = 0 \).
(C) One valuation \( v \) such that \( \Sigma^v = 1 \) and \( A^v = 1 \).
(D) One valuation \( v \) such that \( \Sigma^v = 1 \) and \( A^v = 0 \).
Example: Prove/Disprove a logical consequence

Consider the logical consequence below.

\[ \forall x \neg A(x) \vdash \neg (\exists x A(x)). \]

If the logical consequence holds, prove it.
If it does not hold, provide a counterexample.
Example: Prove/Disprove a logical consequence

Consider the logical consequence below.

$$\forall x(A(x) \rightarrow B(x)) \vdash \forall x A(x) \rightarrow \forall x B(x)$$

If the logical consequence holds, prove it.
If it does not hold, provide a counterexample.
Example: Prove/Disprove a logical consequence

Consider the logical consequence below.

$$\forall x A(x) \rightarrow \forall x B(x) \models \forall x (A(x) \rightarrow B(x))$$

If the logical consequence holds, prove it.
If it does not hold, provide a counterexample.
Example: Prove/Disprove the logical consequence

Prove the following

$$\exists x (A(x) \land B(x)) \vdash \exists x A(x) \land \exists x B(x).$$
Example: Prove/Disprove the logical consequence

Prove the following

$$\exists x A(x) \land \exists x B(x) \not\equiv \exists x (A(x) \land B(x)).$$
Revisiting the learning goals

By the end of this lecture, you should be able to:

- Define logical consequence for predicate logic.
- Prove that a logical consequence holds.
- Prove that a logical consequence does not hold.