Predicate Logic: Semantics

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Lecture 13
Outline

The Learning Goals

Evaluating Terms and Formulas w/o Variables

Evaluating Terms and Formulas w/o Bound Variables

Evaluating Quantified Formulas

A few clarifications

Satisfiable and Valid

Revisiting the Learning Goals
Learning goals

By the end of this lecture, you should be able to:

▶ Define a valuation.
▶ Determine the value of a term given a valuation.
▶ Determine the truth value of a formula given a valuation.
▶ Give a valuation that makes a formula true or false.
▶ Determine and justify whether a formula is satisfiable and/or valid.
The Language of Predicate Logic

- **Domain**: a non-empty set of objects
- **Individuals**: concrete objects in the domain
- **Functions**: takes objects in the domain as arguments and returns an object of the domain.
- **Relations**: takes objects in the domain as arguments and returns true or false. They describe properties of objects or relationships between objects.
- **Variables**: placeholders for concrete objects in the domain
- **Quantifiers**: for how many objects in the domain is the statement true?
The semantics of a predicate formula

Given a well-formed formula of predicate logic, does the formula evaluate to 0 or 1 in some context?

Example: What does $(F(a) \lor G(a, b))$ mean?

The symbols $F, G, a,$ and $b$ do not have intrinsic meanings.

In propositional logic, we need a truth valuation to give a meaning to a formula.

In predicate logic, we need a valuation to give a meaning to a term or a formula.
A valuation $v$ for our language $\mathcal{L}$ consists of

1. A domain $D$,
2. A meaning for each individual symbol, e.g. $a^v \in D$,
3. A meaning for each free variable symbol, e.g. $u^v \in D$,
4. A meaning for each relation symbol, e.g. $F^v \subseteq D^n$, $\approx^v = \{ \langle x, x \rangle \mid x \in D \} \subseteq D^2$.
5. A meaning for each function symbol, e.g. $f^v : D^m \rightarrow D$. 
A function symbol must be interpreted as a total function

A function symbol $f$ must be interpreted as a function $f^v$ that is total on the domain $D$.

$$f^v : D^m \rightarrow D$$

- Any $m$-tuple $(d_1, \ldots, d_m) \in D^m$ can be an input to $f^v$.
- For any legal $m$-tuple $(d_1, \ldots, d_m) \in D^m$, $f^v(d_1^v, \ldots, d_m^v) \in D$. 
CQ Which function is total?

Which of the following functions is total?

(A) $g(x, y) = x - y$. $D = \mathbb{N}$ (natural numbers).
   Incorrect. $1 - 2 = -1$ is not a natural number.

(B) $f(x) = \sqrt{x}$. $D = \mathbb{Z}$ (integers).
   Incorrect. The square root of an integer may not be an integer anymore.

(C) $f(x) = x + 1$. $D = \{1, 2, 3\}$.
   Incorrect. $3 + 1 = 4$ not in domain.

(D) $f(1) = 2$, $f(2) = 3$ and $f(3) = 3$. $D = \{1, 2, 3\}$.
   Correct

(E) $g(x, y) = x > y$. $D = \mathbb{Z}$ (integers).
   Incorrect. $x > y$ produces true or false, not a domain element.
Definition (Value of Terms)

The value of terms of $L$ under valuation $v$ over domain $D$ is defined by recursion:

1. $a^v \in D$.
2. $u^v \in D$.
3. $f(t_1, \ldots, t_n)^v = f^v(t_1^v, \ldots, t_n^v)$. 
The assignment override notation

\( v(u/\alpha) \) keeps all the mappings in \( v \) intact EXCEPT reassigning \( u \) to \( \alpha \in D \).

Consider a valuation: \( u^v_1 = 3, \ u^v_2 = 3, \ u^v_3 = 1 \). \( D = \{1, 2, 3\} \).

1. \( u^v_1 u_1^v(u_1/2) = ? \)
2. \( u^v_2 u_2^v(u_1/2) = ? \)
3. \( u^v_1 u_1^v(u_1/2)(u_2/1) = ? \)
4. \( u^v_2 u_2^v(u_1/2)(u_2/1) = ? \)
5. \( u^v_3 u_3^v(u_1/2)(u_2/1) = ? \)

\( u^v_1 u_1^v(u_1/2) = 2, \ u^v_2 u_2^v(u_1/2) = 3, \ u^v_1 u_1^v(u_1/2)(u_2/1) = 2, \ u^v_2 u_2^v(u_1/2)(u_2/1) = 1, \ u^v_3 u_3^v(u_1/2)(u_2/1) = 1 \)
True Value of Formulas

**Definition (Truth Value of Formulas)**

The truth value of formulas of $L$ under valuation $v$ over domain $D$ is defined by recursion:

1. $F(t_1, \ldots, t_n)^v = 1$ iff $\langle t_1^v, \ldots, t_n^v \rangle \in F^v$.
2. $(\neg A)^v = 1$ iff $A^v = 0$.
3. $(A \land B)^v = 1$ iff $A^v = 1$ and $B^v = 1$.
4. $(A \lor B)^v = 1$ iff $A^v = 1$ or $B^v = 1$.
5. $(A \rightarrow B)^v = 1$ iff $A^v = 0$ or $B^v = 1$.
6. $(A \leftrightarrow B)^v = 1$ iff $A^v = B^v$.
7. $(\forall x \ A(x))^v = 1$ iff for every $\alpha \in D$, $A(u)^v(u/\alpha) = 1$, where $u$ does not occur in $A(x)$.
8. $(\exists x \ A(x))^v = 1$ iff there exists $\alpha \in D$, $A(u)^v(u/\alpha) = 1$, where $u$ does not occur in $A(x)$. 

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Our language of predicate logic:

Individual symbols: $a, b, c$.
Free variable symbols: $u, v, w$.
Bound variable symbols: $x, y, z$.
Function symbols: $f$ is a unary function. $g$ is a binary function.
Relation symbols: $F$ is a unary relation. $G$ is a binary relation.
An example of a valuation

Valuation \( v \):

- **Domain**: \( D = \{1, 2, 3\} \).
- **Individuals**: \( a^v = 1, b^v = 2, c^v = 3 \).
- **Free variables**: \( u^v = 3, v^v = 2, w^v = 1 \).
- **Functions**:
  \[
  f^v: \quad f^v(1) = 2, f^v(2) = 3, f^v(3) = 1.
  \]
  \[
  g^v: \quad g^v(x, y) = ((x + y) \mod 3) + 1.
  \]
- **Relations**:
  \[
  F^v: \quad F^v(x) \text{ is true if and only if } x > 5.
  \]
  \[
  G^v: \quad G^v(x, y) \text{ is true if and only if } x > y.
  \]
Another example of a valuation

Valuation $v'$:

- **Domain:** $D = \{Alice, Bob, Cate\}$.
- **Individuals:** $a^v = Alice, b^v = Bob, c^v = Cate$.
- **Free variables:** $u^v = Bob, v^v = Alice, w^v = Alice$.
- **Functions:**
  - $f^v: f^v(Alice) = Alice, f^v(Bob) = Cate, f^v(Cate) = Bob$.
  - $g^v: g^v(x, y) = $ the person with the longer name. return $x$ if there is a tie.
- **Relations:**
  - $F^v: F^v(x)$ is true iff the person likes chocolates. (Alice and Cate like chocolates whereas Bob dislikes chocolates.)
  - $G^v: G^v(x, y)$ is true iff $x$ is older than or has the same age as $y$. (Alice is older than Cate, who is older than Bob.)
Consider the domain $D = \{1, 2, 3\}$.

Functions:

- $f^v$ is the identify function. $f^v(x) = x$.
- $f^v(1) = 1$, $f^v(2) = 2$ and $f^v(3) = 3$.

Relations:

- $G^v$: $G^v(x, y)$ is true if and only if $x > y$.
- $G^v = \{\langle 2, 1 \rangle, \langle 3, 1 \rangle, \langle 3, 2 \rangle \}$
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A few clarifications

Satisfiable and Valid

Revisiting the Learning Goals
Evaluating terms and formulas w/o variables

Evaluate these terms and formulas under the valuation \( v \).
\( f(f(a)) \), \( (F(a) \lor G(a, b)) \).

Valuation \( v \):

- **Domain:** \( D = \{1, 2, 3\} \).
- **Individuals:** \( a^v = 1 \), \( b^v = 2 \), \( c^v = 3 \).
- **Free variables:** \( u^v = 3 \), \( v^v = 2 \), \( w^v = 1 \).
- **Functions:**
  \( f^v \):
  \[
  f^v(1) = 2, \quad f^v(2) = 3, \quad f^v(3) = 1.
  \]
  \( g^v \):
  \[
  g^v(x, y) = ((x + y) \mod 3) + 1.
  \]
- **Relations:**
  \( F^v \):
  \( F^v(x) \) is true if and only if \( x > 5 \).
  \( G^v \):
  \( G^v(x, y) \) is true if and only if \( x > y \).
\[
\begin{align*}
  f(f(a))^v &= ?, \quad a^v = 1, \quad f^v(a^v) = 2,
  f^v(f^v(a^v)) = 3 \\
  (F(a) \lor G(a, b))^v &= ?, \quad F(a)^v = 0, \quad G(a, b)^v = 0,
  (F(a) \lor G(a, b))^v = 0
\end{align*}
\]
Give a valuation that makes the formula true/false

Complete the valuation \( v \) such that

(A) \( G(a, f(f(a)))^v = 1 \)
(B) \( G(a, f(f(a)))^v = 0 \)

Valuation \( v \):

- Domain: \( D = \{1, 2, 3\} \).
- Individuals: \( a^v = ?, b^v = ?, c^v = ? \).
- Functions: \( f^v : ?, g^v : ? \)
- Relations: \( P^v : ?, G^v : ? \)

\( a^v = 1 \)
\( f^v \) is the identity function.

To make the formula true, make sure \( \langle 1, 1 \rangle \in G \).
To make the formula false, make sure \( \langle 1, 1 \rangle \notin G \).
If \( G^v = \emptyset \), the formula is false.
If \( G^v \) contains all possible tuples, the formula is true.
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Revisiting the Learning Goals
A valuation for interpreting free variables

Valuation \( v \):

- **Domain**: \( D = \{1, 2, 3\} \).
- **Individuals**: \( a^v = 1 \), \( b^v = 2 \), \( c^v = 3 \).
- **Free variables**: \( u^v = 3 \), \( v^v = 2 \), \( w^v = 1 \).
- **Functions**:
  - \( f^v \): \( f^v(1) = 2 \), \( f^v(2) = 3 \), \( f^v(3) = 1 \).
  - \( g^v \): \( g^v(x, y) = ((x + y) \mod 3) + 1 \).
- **Relations**:
  - \( F^v \): \( F^v(x) \) is true if and only if \( x > 5 \).
  - \( G^v \): \( G^v(x, y) \) is true if and only if \( x > y \).
Evaluating terms & formulas w/o bound variables

Evaluate these terms and formulas under the valuation $v$.

$g(u, f(b)), G(a, f(f(u)))$.

Valuation $v$:

- **Domain:** $D = \{1, 2, 3\}$.
- **Individuals:** $a^v = 1$, $b^v = 2$, $c^v = 3$.
- **Free variables:** $u^v = 3$, $v^v = 2$, $w^v = 1$.
- **Functions:**
  - $f^v$: $f^v(1) = 2$, $f^v(2) = 3$, $f^v(3) = 1$.
  - $g^v$: $g^v(x, y) = ((x + y) \mod 3) + 1$.
- **Relations:**
  - $F^v$: $F^v(x)$ is true if and only if $x > 5$.
  - $G^v$: $G^v(x, y)$ is true if and only if $x > y$.

$g(x, f(b))^v = 1$.

$G(a, f(f(x)))^v = 0$. 
Give a valuation that makes the formula true/false

Complete a valuation such that

(A) \( G(a, f(f(u)))^v = 1 \)
(B) \( G(a, f(f(u)))^v = 0 \)

Valuation \( v \):

- Domain: \( D = \{1, 2, 3\} \).
- Individuals: \( a^v = ?, b^v = ?, c^v = ? \).
- Free variables: \( u^v = ?, u^v = ? \).
- Functions: \( f^v : ?, g^v : ? \).
- Relations: \( P^v : ?, G^v : ? \).

\( x^v = 1 \). \( f^v \) is the identity function. \( a^v = 1 \).

To make \( G \) true, let \( \langle 1, 1 \rangle \in G^v \).
To make \( G \) true, let \( \langle 1, 1 \rangle \notin G^v \).
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Revisiting the Learning Goals
Evaluate quantified formulas under a valuation

Evaluate these formulas under the valuation \( v \).

(A) \( \forall x (\exists y \, G(x, y)) \)

(B) \( \exists x (\forall y \, G(x, y)) \)

Valuation \( v \):

- Domain: \( D = \{1, 2, 3\} \).
- Relations: \( G^v = \{\langle 1, 2\rangle, \langle 3, 1\rangle, \langle 2, 3\rangle\} \).
(A) \((\forall x (\exists y \ G(x,y)))\)
(B) \((\exists x (\forall y \ G(x,y)))\)

\[
\langle u^v(u/1)(w/2), w^v(u/1)(w/2) \rangle = \langle 1, 2 \rangle \in G^v,
\langle u^v(u/3)(w/1), w^v(u/3)(w/1) \rangle = \langle 3, 1 \rangle \in G^v,
\langle u^v(u/2)(w/3), w^v(u/2)(w/3) \rangle = \langle 2, 3 \rangle \in G^v,
\]

Therefore, \((\forall x (\exists y \ G(x,y)))^v = 1\).

\[
\langle u^v(u/1)(w/1), w^v(u/1)(w/1) \rangle = \langle 1, 1 \rangle \notin G^v,
\langle u^v(u/3)(w/2), w^v(u/3)(w/2) \rangle = \langle 3, 2 \rangle \notin G^v,
\langle u^v(u/2)(w/1), w^v(u/2)(w/1) \rangle = \langle 2, 1 \rangle \notin G^v,
\]

Therefore, \((\exists x (\forall y \ G(x,y)))^v = 0\).
Give a valuation that makes the formula true/false

Complete the valuation \( v \) to make the following formula true/false. (When satisfying the formula, try making \( G^v \) as small as possible.)

(A) \( (\forall y \, (\exists x \, G(x, y))) \)
(B) \( (\exists y \, (\forall x \, G(x, y))) \)

Valuation \( v \):

- Domain: \( D = \{1, 2, 3\} \).
- ...

(A) \( G^v = \{\langle 1, 1 \rangle, \langle 1, 2 \rangle, \langle 1, 3 \rangle\} \) makes the formula true.
(B) \( G^v = \{\langle 1, 1 \rangle, \langle 2, 1 \rangle, \langle 3, 1 \rangle\} \) makes the formula true.
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Revisiting the Learning Goals
Difference between Individual and Free Variable Symbols

Let our domain be the set of people. Let the predicate $L(u)$ be true if $u$ likes chocolates. Let $a$ be an individual symbol referring to Alice.

- $L(a)$
  This formula only contains individual symbols. Since $a$ refers to Alice, the truth value of this formula is already determined (It’s true because Alice likes chocolates =).

- $L(u)$
  This formula only contains free variable symbols. We do not know the truth value of this formula because $u$ can refer to any person in the domain. We need to assign $u$ to a particular person because we can determine whether this formula is true or false.
Let our domain be the set of integers.

- $u + u = v$
  The variables are free.
  We do not know the truth value of this formula until we assign the free variables to elements of the domain.

- $\forall x \forall y (x + x = y)$
  The variables are bound.
  We know the truth value of this formula because the meanings of the variables are given by the quantifiers.
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Satisfiable and Valid

Revisiting the Learning Goals
A formula $A$ is **satisfiable:**

there exists a valuation $v$, $A^v = 1$.

A formula $A$ is **valid:**

for every valuation $v$, $A^v = 1$.

Most predicate formulas are satisfiable but not valid because we have a great deal of freedom to choose the valuation.
Is the following formula satisfiable? If it’s satisfiable, give a valuation that satisfies it. If it’s not satisfiable, give a proof.

\[(\exists x \ F(x)) \rightarrow (\forall x \ F(x))\]

Answer: This formula is satisfiable.

Proof:
Consider the valuation: \(D = \{1, 2\}\), and \(F^v = \emptyset\).
Since \(F^v = \emptyset\), \(F(u)^v(u/\alpha) = 0\) for every \(\alpha \in D\).
Thus, \((\exists x \ F(x))^v = 0\) and \(((\exists x \ F(x)) \rightarrow (\forall x \ F(x)))^v = 1\).

Here is another valuation that works: \(D = \{1, 2\}\) and \(F^v = \{1, 2\}\).
Proving that a formula is valid/not valid

Is the following formula valid? If it’s valid, give a proof. If it’s not valid, give a counterexample.

\[(\forall x \ F(x)) \rightarrow (\exists x \ F(x))\]

- Determine whether the formula is valid or not.

- How do I prove that a formula is NOT valid?
  Find a valuation under which the formula is false.
  In this case, find a valuation under which \( \forall x \ F(x) \) is true and \( \exists x \ F(x) \) is false.

- How do I prove that a formula is valid?
  Consider any valuation. Prove that the formula must be true.
  In this case, consider any valuation \( v \) under which \( \forall x \ F(x) \) is true. Show that \( \exists x \ F(x) \) is also true under \( v \).
Proving that a formula is valid or not

Is the following formula valid? If it’s valid, give a proof. If it’s not valid, give a counterexample.

\[(\forall x \ F(x)) \rightarrow (\exists x \ F(x))\]

Answer: This formula is valid. 

Proof: 
We prove this by contradiction. Assume that there is a valuation \(v\) such that \((\forall x F(x))^v = 1\) and \((\exists x F(x))^v = 0\). Form \(F(u)\) from \(F(x)\), \(u\) not occurring in \(F(x)\). 

Since \((\forall x F(x))^v = 1\), then for every \(\alpha \in D\), \(F(u)^v(u/\alpha) = 1\). (1) 

Since \((\exists x F(x))^v = 0\), then there exists \(\alpha \in D\), \(F(u)^v(u/\alpha) = 0\), which contradicts (1). 
QED
Proving that a formula is valid or not

Is the following formula valid? If it’s valid, give a proof. If it’s not valid, give a counterexample.

\[(\exists x \ F(x)) \rightarrow (\forall x \ F(x))\]

Answer: This formula is not valid.
Proof:
Consider the valuation below.

- \(D = \{1, 2\}\)
- \(F^v = \{1\}\)

Since \(1 \in F^v\), \(F(u)^v(u/1) = 1\). Thus, \((\exists x \ F(x))^v = 1\).
Since \(2 \notin F^v\), \(F(u)^v(u/2) = 0\). Thus, \((\forall x \ F(x))^v = 0\).
Revisiting the learning goals

By the end of this lecture, you should be able to:

▶ Define a valuation.
▶ Determine the value of a term given a valuation.
▶ Determine the truth value of a formula given a valuation.
▶ Give a valuation that makes a formula true or false.
▶ Determine and justify whether a formula is satisfiable and/or valid.