Predicate Logic: Semantics

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Lecture 13

Outline

Learning goals

By the end of this lecture, you should be able to:

- Define a valuation.
- Determine the value of a term given a valuation.
- Determine the truth value of a formula given a valuation.
- Give a valuation that makes a formula true or false.
- Determine and justify whether a formula is satisfiable and/or valid.

The Language of Predicate Logic

- Domain: a non-empty set of objects
- Individuals: concrete objects in the domain
- Functions: takes objects in the domain as arguments and returns an object of the domain.
- Relations: takes objects in the domain as arguments and returns true or false. They describe properties of objects or relationships between objects.
- Variables: placeholders for concrete objects in the domain
- Quantifiers: for how many objects in the domain is the statement true?

The semantics of a predicate formula

Given a well-formed formula of predicate logic,
does the formula evaluate to 0 or 1 in some context?

Example: What does $(F(a) \vee G(a, b))$ mean?

The symbols F, G, a, and b do not have intrinsic meanings.

In propositional logic, we need a truth valuation
to give a meaning to a formula.

In predicate logic, we need a valuation
to give a meaning to a term or a formula.

Valuation

A valuation v for our language $\mathcal{L}$ consists of

1. A domain D,
2. A meaning for each individual symbol, e.g. $a^v \in D$,
3. A meaning for each free variable symbol, e.g. $u^v \in D$,
4. A meaning for each relation symbol, e.g. $F^v \subseteq D^n$,
   $\approx^v = \{\langle x, x \rangle\} x \in D\} \subseteq D^2$.
5. A meaning for each function symbol, e.g. $f^v : D^m \to D$.

A function symbol must be interpreted as a total function

A function symbol f must be interpreted as a function f^v that is total on the domain D.

$$f^v : D^m \to D$$

- Any m-tuple $(d_1, ..., d_m) \in D^m$ can be an input to f^v.
- For any legal m-tuple $(d_1, ..., d_m) \in D^m$, $f^v(d_1^v, ..., d_m^v) \in D$.

CQ Which function is total?

Which of the following functions is total?

(A) $g(x, y) = x - y$. $D = \mathbb{N}$ (natural numbers).
Incorrect. 1 - 2 = -1 is not a natural number.

(B) $f(x) = \sqrt{x}$. $D = \mathbb{Z}$ (integers).
Incorrect. The square root of an integer may not be an integer anymore.

(C) $f(x) = x + 1$. $D = \{1, 2, 3\}$.
Incorrect. 3 + 1 = 4 not in domain.

(D) $f(1) = 2$, $f(2) = 3$ and $f(3) = 3$. $D = \{1, 2, 3\}$.
Correct

(E) $g(x, y) = x > y$. $D = \mathbb{Z}$ (integers).
Incorrect. x > y produces true or false, not a domain element.

Value of Terms

Definition (Value of Terms)

The value of terms of L under valuation v over domain D is defined by recursion:

1. $a^v \in D$.
2. $u^v \in D$.
3. $f(t_1, \ldots, t_n)^v = f^v(t_1^v, \ldots, t_n^v)$.

The assignment override notation

$v(u/\alpha)$ keeps all the mappings in v intact
EXCEPT reassigning u to $\alpha \in D$.

Consider a valuation: $u_1^v = 3$, $u_2^v = 3$, $u_3^v = 1$. $D = \{1, 2, 3\}$.

1. $u_1^{v(u_1/2)} = ?$
2. $u_2^{v(u_1/2)} = ?$
3. $u_1^{v(u_1/2)(u_2/1)} = ?$
4. $u_2^{v(u_1/2)(u_2/1)} = ?$
5. $u_3^{v(u_1/2)(u_2/1)} = ?$

$u_1^{v(u_1/2)} = 2$, $u_2^{v(u_1/2)} = 3$
$u_1^{v(u_1/2)(u_2/1)} = 2$, $u_2^{v(u_1/2)(u_2/1)} = 1$, $u_3^{v(u_1/2)(u_2/1)} = 1$

True Value of Formulas

Definition (Truth Value of Formulas)

The truth value of formulas of L under valuation v over domain D is defined by recursion:

1. $F(t_1, \dots, t_n)^v = 1$ iff $\langle t_1^v, \dots, t_n^v \rangle \in F^v$.
2. $(\neg A)^v = 1$ iff $A^v = 0$.
3. $(A \land B)^v = 1$ iff $A^v = 1$ and $B^v = 1$.
4. $(A \lor B)^v = 1$ iff $A^v = 1$ or $B^v = 1$.
5. $(A \to B)^v = 1$ iff $A^v = 0$ or $B^v = 1$.
6. $(A \leftrightarrow B)^v = 1$ iff $A^v = B^v$.
7. $(\forall x\ A(x))^v = 1$ iff for every $\alpha \in D$, $A(u)^{v(u/\alpha)} = 1$, where u does not occur in $A(x)$.
8. $(\exists x\ A(x))^v = 1$ iff there exists $\alpha \in D$, $A(u)^{v(u/\alpha)} = 1$, where u does not occur in $A(x)$.

Our predicate logic language

Our language of predicate logic:

Individual symbols: a, b, c.
Free variable symbols: u, v, w.
Bound variable symbols: x, y, z.
Function symbols: f is a unary function. g is a binary function.
Relation symbols: F is a unary relation. G is a binary relation.

An example of a valuation

Valuation v:

- Domain: $D = \{1, 2, 3\}$.
- Individuals: $a^v = 1$, $b^v = 2$, $c^v = 3$.
- Free variables: $u^v = 3$, $v^v = 2$, $w^v = 1$.
- Functions:
  f^v: $f^v(1) = 2, f^v(2) = 3, f^v(3) = 1$.
  g^v: $g^v(x, y) = ((x + y) \mod 3) + 1$.
- Relations:
  F^v: $F^v(x)$ is true if and only if $x > 5$.
  G^v: $G^v(x, y)$ is true if and only if $x > y$.

Another example of a valuation

Valuation v':

- Domain: $D = \{Alice, Bob, Cate\}$.
- Individuals: $a^v = Alice$, $b^v = Bob$, $c^v = Cate$.
- Free variables: $u^v = Bob$, $v^v = Alice$, $w^v = Alice$.
- Functions:
  f^v: $f^v(Alice) = Alice, f^v(Bob) = Cate, f^v(Cate) = Bob$.
  g^v: $g^v(x, y) =$ the person with the longer name. return x if there is a tie.
- Relations:
  F^v: $F^v(x)$ is true iff the person likes chocolates. (Alice and Cate like chocolates whereas Bob dislikes chocolates.)
  G^v: $G^v(x, y)$ is true iff x is older than or has the same age as y. (Alice is older than Cate, who is older than Bob.)

Notation for functions and relations

Consider the domain $D = \{1, 2, 3\}$.

Functions:

- f^v is the identify function. $f^v(x) = x$.
- $f^v(1) = 1$, $f^v(2) = 2$ and $f^v(3) = 3$.

Relations:

- G^v: $G^v(x, y)$ is true if and only if $x > y$.
- $G^v = \{\langle 2, 1\rangle, \langle 3, 1\rangle, \langle 3, 2\rangle\}$

Outline

Evaluating terms and formulas w/o variables

Evaluate these terms and formulas under the valuation v.
$f(f(a))$, $(F(a) \vee G(a, b))$.

Valuation v:

- Domain: $D = \{1, 2, 3\}$.
- Individuals: $a^v = 1$, $b^v = 2$, $c^v = 3$.
- Free variables: $u^v = 3$, $v^v = 2$, $w^v = 1$.
- Functions:
  f^v: $f^v(1) = 2, f^v(2) = 3, f^v(3) = 1$.
  g^v: $g^v(x, y) = ((x + y) \mod 3) + 1$.
- Relations:
  F^v: $F^v(x)$ is true if and only if $x > 5$.
  G^v: $G^v(x, y)$ is true if and only if $x > y$.

$f(f(a))^v = ?$, $a^v = 1$, $f^v(a^v) = 2$, $f^v(f^v(a^v)) = 3$
$(F(a) \vee G(a, b))^v = ?$, $F(a)^v = 0$, $G(a, b)^v = 0$,
$(F(a) \vee G(a, b))^v = 0$

Give a valuation that makes the formula true/false

Complete the valuation v such that

(A) $G(a, f(f(a)))^v = 1$

(B) $G(a, f(f(a)))^v = 0$

Valuation v:

- Domain: $D = \{1, 2, 3\}$.
- Individuals: $a^v = ?$, $b^v = ?$, $c^v = ?$.
- Functions: $f^v : ?$, $g^v : ?$
- Relations: $P^v : ?$, $G^v : ?$

$a^v = 1$
f^v is the identity function.
To make the formula true, make sure $\langle 1, 1 \rangle \in G$.
To make the formula false, make sure $\langle 1, 1 \rangle \notin G$.
If $G^v = \emptyset$, the formula is false.
If G^v contains all possible tuples, the formula is true.

Outline

A valuation for interpreting free variables

Valuation v:

- Domain: $D = \{1, 2, 3\}$.
- Individuals: $a^v = 1$, $b^v = 2$, $c^v = 3$.
- Free variables: $u^v = 3$, $v^v = 2$, $w^v = 1$.
- Functions:
  f^v: $f^v(1) = 2, f^v(2) = 3, f^v(3) = 1$.
  g^v: $g^v(x, y) = ((x + y) \mod 3) + 1$.
- Relations:
  F^v: $F^v(x)$ is true if and only if $x > 5$.
  G^v: $G^v(x, y)$ is true if and only if $x > y$.

Evaluating terms & formulas w/o bound variables

Evaluate these terms and formulas under the valuation v.
$g(u, f(b))$, $G(a, f(f(u)))$.

Valuation v:

- Domain: $D = \{1, 2, 3\}$.
- Individuals: $a^v = 1$, $b^v = 2$, $c^v = 3$.
- Free variables: $u^v = 3$, $v^v = 2$, $w^v = 1$.
- Functions:
  f^v: $f^v(1) = 2, f^v(2) = 3, f^v(3) = 1$.
  g^v: $g^v(x, y) = ((x + y) \mod 3) + 1$.
- Relations:
  F^v: $F^v(x)$ is true if and only if $x > 5$.
  G^v: $G^v(x, y)$ is true if and only if $x > y$.

$g(u, f(b))^v = 1$.
$G(a, f(f(u)))^v = 0$.

Give a valuation that makes the formula true/false

Complete a valuation such that

(A) $G(a, f(f(u)))^v = 1$

(B) $G(a, f(f(u)))^v = 0$

Valuation v:

- Domain: $D = \{1, 2, 3\}$.
- Individuals: $a^v = ?$, $b^v = ?$, $c^v = ?$.
- Free variables: $u^v = ?$, $u^v = ?$ $u^v = ?$.
- Functions: $f^v : ?$, $g^v : ?$
- Relations: $P^v : ?$, $G^v : ?$

$u^v = 1$. f^v is the identity function. $a^v = 1$.
To make G true, let $\langle 1, 1 \rangle \in G^v$.
To make G true, let $\langle 1, 1 \rangle \notin G^v$.

Outline

Evaluate quantified formulas under a valuation

Evaluate these formulas under the valuation v.

(A) $(\forall x\ (\exists y\ G(x,y)))$

(B) $(\exists x\ (\forall y\ G(x,y)))$

Valuation v:

- Domain: $D = \{1, 2, 3\}$.
- Relations: $G^v = \{\langle 1, 2\rangle, \langle 3, 1\rangle, \langle 2, 3\rangle\}$.

(A) $(\forall x\ (\exists y\ G(x,y)))$
(B) $(\exists x\ (\forall y\ G(x,y)))$

$\langle u^{v(u/1)(w/2)}, w^{v(u/1)(w/2)} \rangle = \langle 1, 2 \rangle \in G^v$,
$\langle u^{v(u/3)(w/1)}, w^{v(u/3)(w/1)} \rangle = \langle 3, 1 \rangle \in G^v$,
$\langle u^{v(u/2)(w/3)}, w^{v(u/2)(w/3)} \rangle = \langle 2, 3 \rangle \in G^v$,
Therefore, $(\forall x\ (\exists y\ G(x,y)))^v = 1$.

$\langle u^{v(u/1)(w/1)}, w^{v(u/1)(w/1)} \rangle = \langle 1, 1 \rangle \notin G^v$,
$\langle u^{v(u/3)(w/2)}, w^{v(u/3)(w/2)} \rangle = \langle 3, 2 \rangle \notin G^v$,
$\langle u^{v(u/2)(w/1)}, w^{v(u/2)(w/1)} \rangle = \langle 2, 1 \rangle \notin G^v$,
Therefore, $(\exists x\ (\forall y\ G(x,y)))^v = 0$.

Give a valuation that makes the formula true/false

Complete the valuation v to make the following formula true/false. (When satisfying the formula, try making G^v as small as possible.)

(A) $(\forall y\ (\exists x\ G(x, y)))$

(B) $(\exists y\ (\forall x\ G(x, y)))$

Valuation v:

- Domain: $D = \{1, 2, 3\}$.
- ...

(A) $G^v = \{\langle 1, 1\rangle, \langle 1, 2\rangle, \langle 1, 3\rangle\}$ makes the formula true.
(B) $G^v = \{\langle 1, 1\rangle, \langle 2, 1\rangle, \langle 3, 1\rangle\}$ makes the formula true.

Outline

Difference between Individual and Free Variable Symbols

Let our domain be the set of people. Let the predicate $L(u)$ be true if u likes chocolates. Let a be an individual symbol referring to Alice.

- $L(a)$
  This formula only contains individual symbols.
  Since a refers to Alice, the truth value of this formula is already determined (It's true because Alice likes chocolates $=$).
- $L(u)$
  This formula only contains free variable symbols.
  We do not know the truth value of this formula because u can refer to any person in the domain. We need to assign u to a particular person because we can determine whether this formula is true or false.

Difference between Free and Bound Variables

Let our domain be the set of integers.

- $u + u = v$
  The variables are free.
  We do not know the truth value of this formula until we assign the free variables to elements of the domain.
- $\forall x \, \forall y \, (x + x = y)$
  The variables are bound.
  We know the truth value of this formula because the meanings of the variables are given by the quantifiers.

Outline

Satisfiable and Valid

A formula A is satisfiable:

there exists a valuation v, $A^v = 1$.

A formula A is valid:

for every valuation v, $A^v = 1$.

Most predicate formulas are satisfiable but not valid
because we have a great deal of freedom to choose the valuation.

Proving that a formula is satisfiable or not

Is the following formula satisfiable? If it's satisfiable, give a valuation that satisfies it. If it's not satisfiable, give a proof.

$$(\exists x\ F(x)) \rightarrow (\forall x\ F(x))$$

Answer: This formula is satisfiable.
Proof:
Consider the valuation: $D = \{1, 2\}$, and $F^v = \emptyset$.
Since $F^v = \emptyset$, $F(u)^{v(u/\alpha)} = 0$ for every $\alpha \in D$.
Thus, $(\exists x\ F(x))^v = 0$ and $((\exists x\ F(x)) \rightarrow (\forall x\ F(x)))^v = 1$.

Here is another valuation that works: $D = \{1, 2\}$ and $F^v = \{1, 2\}$.

Proving that a formula is valid/not valid

Is the following formula valid? If it's valid, give a proof.
If it's not valid, give a counterexample.

$$(\forall x \; F(x)) \to (\exists x \; F(x))$$

- Determine whether the formula is valid or not.
- How do I prove that a formula is NOT valid?
  Find a valuation under which the formula is false.
  In this case, find a valuation under which $\forall x \; F(x)$ is true and $\exists x \; F(x)$ is false.
- How do I prove that a formula is valid?
  Consider any valuation. Prove that the formula must be true.
  In this case, consider any valuation v under which $\forall x \; F(x)$ is true. Show that $\exists x \; F(x)$ is also true under v.

Proving that a formula is valid or not

Is the following formula valid? If it's valid, give a proof.
If it's not valid, give a counterexample.

$$(\forall x \ F(x)) \rightarrow (\exists x \ F(x))$$

Answer: This formula is valid.
Proof:
We prove this by contradiction. Assume that there is a valuation v such that $(\forall x F(x))^v = 1$ and $(\exists x F(x))^v = 0$. Form $F(u)$ from $F(x)$, u not occurring in $F(x)$.
Since $(\forall x F(x))^v = 1$, then for every $\alpha \in D$, $F(u)^{v(u/\alpha)} = 1$. (1)
Since $(\exists x F(x))^v = 0$, then there exists $\alpha \in D$, $F(u)^{v(u/\alpha)} = 0$, which contradicts (1).
QED

Proving that a formula is valid or not

Is the following formula valid? If it's valid, give a proof.
If it's not valid, give a counterexample.

$$(\exists x\ F(x)) \to (\forall x\ F(x))$$

Answer: This formula is not valid.
Proof:
Consider the valuation below.

- $D = \{1, 2\}$
- $F^v = \{1\}$

Since $1 \in F^v$, $F(u)^{v(u/1)} = 1$. Thus, $(\exists x\ F(x))^v = 1$.
Since $2 \notin F^v$, $F(u)^{v(u/2)} = 0$. Thus. $(\forall x\ F(x))^v = 0$.

Revisiting the learning goals

By the end of this lecture, you should be able to:

- Define a valuation.
- Determine the value of a term given a valuation.
- Determine the truth value of a formula given a valuation.
- Give a valuation that makes a formula true or false.
- Determine and justify whether a formula is satisfiable and/or valid.