Predicate Logic: Semantics

Alice Gao

Lecture 13
Outline

The Learning Goals

Evaluating Terms and Formulas w/o Variables

Evaluating Terms and Formulas w/o Bound Variables

Evaluating Quantified Formulas

A few clarifications

Satisfiable and Valid

Revisiting the Learning Goals
Learning goals

By the end of this lecture, you should be able to:

▶ Define a valuation.
▶ Determine the value of a term given a valuation.
▶ Determine the truth value of a formula given a valuation.
▶ Give a valuation that makes a formula true or false.
▶ Determine and justify whether a formula is satisfiable and/or valid.
The Language of Predicate Logic

- Domain: a non-empty set of objects
- Individuals: concrete objects in the domain
- Functions: takes objects in the domain as arguments and returns an object of the domain.
- Relations: takes objects in the domain as arguments and returns true or false. They describe properties of objects or relationships between objects.
- Variables: placeholders for concrete objects in the domain
- Quantifiers: for how many objects in the domain is the statement true?
The semantics of a predicate formula

Given a well-formed formula of predicate logic, does the formula evaluate to 0 or 1 in some context?

Example: What does \((F(a) \lor G(a, b))\) mean?

The symbols \(F, G, a,\) and \(b\) do not have intrinsic meanings.

In propositional logic, we need a truth valuation to give a meaning to a formula.

In predicate logic, we need a valuation to give a meaning to a term or a formula.
A valuation $\nu$ for our language $\mathcal{L}$ consists of

1. A domain $D$,
2. A meaning for each individual symbol, e.g. $a^\nu \in D$,
3. A meaning for each free variable symbol, e.g. $u^\nu \in D$,
4. A meaning for each relation symbol, e.g. $F^\nu \subseteq D^n$,
   \[ \approx^\nu = \{ \langle x, x \rangle \, | \, x \in D \} \subseteq D^2 \],
5. A meaning for each function symbol, e.g. $f^\nu : D^m \rightarrow D$. 
A function symbol must be interpreted as a total function

A function symbol $f$ must be interpreted as a function $f^v$ that is total on the domain $D$.

$$f^v : D^m \rightarrow D$$

- Any $m$-tuple $(d_1, ..., d_m) \in D^m$ can be an input to $f^v$.
- For any legal $m$-tuple $(d_1, ..., d_m) \in D^m$, $f^v(d_1^v, ..., d_m^v) \in D$. 
Which of the following functions is total?

(A) \( g(x, y) = x - y \). \( D = \mathbb{N} \) (natural numbers).
(B) \( f(x) = \sqrt{x} \). \( D = \mathbb{Z} \) (integers).
(C) \( f(x) = x + 1 \). \( D = \{1, 2, 3\} \).
(D) \( f(1) = 2, \ f(2) = 3 \) and \( f(3) = 3 \). \( D = \{1, 2, 3\} \).
(E) \( g(x, y) = x > y \). \( D = \mathbb{Z} \) (integers).
Value of Terms

Definition (Value of Terms)

The value of terms of $L$ under valuation $v$ over domain $D$ is defined by recursion:

1. $a^v \in D$.
2. $u^v \in D$.
3. $f(t_1, \ldots, t_n)^v = f^v(t_1^v, \ldots, t_n^v)$. 
The assignment override notation

$v(u/\alpha)$ keeps all the mappings in $v$ intact EXCEPT reassigning $u$ to $\alpha \in D$.

Consider a valuation: $u_1^v = 3$, $u_2^v = 3$, $u_3^v = 1$. $D = \{1, 2, 3\}$.

1. $u_1^v(u_1/2) = ?$
2. $u_2^v(u_1/2) = ?$
3. $u_1^v(u_1/2)(u_2/1) = ?$
4. $u_2^v(u_1/2)(u_2/1) = ?$
5. $u_3^v(u_1/2)(u_2/1) = ?$
True Value of Formulas

Definition (Truth Value of Formulas)

The truth value of formulas of \( L \) under valuation \( v \) over domain \( D \) is defined by recursion:

1. \( F(t_1, \ldots, t_n)^v = 1 \) iff \( \langle t_1^v, \ldots, t_n^v \rangle \in F^v \).
2. \( (\neg A)^v = 1 \) iff \( A^v = 0 \).
3. \( (A \land B)^v = 1 \) iff \( A^v = 1 \) and \( B^v = 1 \).
4. \( (A \lor B)^v = 1 \) iff \( A^v = 1 \) or \( B^v = 1 \).
5. \( (A \rightarrow B)^v = 1 \) iff \( A^v = 0 \) or \( B^v = 1 \).
6. \( (A \leftrightarrow B)^v = 1 \) iff \( A^v = B^v \).
7. \( (\forall x \ A(x))^v = 1 \) iff for every \( \alpha \in D \), \( A(u)^v(u/\alpha) = 1 \), where \( u \) does not occur in \( A(x) \).
8. \( (\exists x \ A(x))^v = 1 \) iff there exists \( \alpha \in D \), \( A(u)^v(u/\alpha) = 1 \), where \( u \) does not occur in \( A(x) \).
Our language of predicate logic:

Individual symbols: $a, b, c$.
Free variable symbols: $u, v, w$.
Bound variable symbols: $x, y, z$.
Function symbols: $f$ is a unary function. $g$ is a binary function.
Relation symbols: $F$ is a unary relation. $G$ is a binary relation.
An example of a valuation

Valuation $v$:

- **Domain:** $D = \{1, 2, 3\}$.
- **Individuals:** $a^v = 1$, $b^v = 2$, $c^v = 3$.
- **Free variables:** $u^v = 3$, $v^v = 2$, $w^v = 1$.
- **Functions:**
  - $f^v$: $f^v(1) = 2$, $f^v(2) = 3$, $f^v(3) = 1$.
  - $g^v$: $g^v(x, y) = ((x + y) \mod 3) + 1$.
- **Relations:**
  - $F^v$: $F^v(x)$ is true if and only if $x > 5$.
  - $G^v$: $G^v(x, y)$ is true if and only if $x > y$. 
Another example of a valuation

Valuation $v'$:

- **Domain**: $D = \{Alice, Bob, Cate\}$.
- **Individuals**: $a^v = Alice$, $b^v = Bob$, $c^v = Cate$.
- **Free variables**: $u^v = Bob$, $v^v = Alice$, $w^v = Alice$.
- **Functions**:
  - $f^v$: $f^v(Alice) = Alice$, $f^v(Bob) = Cate$, $f^v(Cate) = Bob$.
  - $g^v$: $g^v(x, y) =$ the person with the longer name. return $x$ if there is a tie.
- **Relations**:
  - $F^v$: $F^v(x)$ is true iff the person likes chocolates. (Alice and Cate like chocolates whereas Bob dislikes chocolates.)
  - $G^v$: $G^v(x, y)$ is true iff $x$ is older than or has the same age as $y$. (Alice is older than Cate, who is older than Bob.)
Consider the domain $D = \{1, 2, 3\}$.

Functions:

- $f^v$ is the identify function. $f^v(x) = x$.
- $f^v(1) = 1$, $f^v(2) = 2$ and $f^v(3) = 3$.

Relations:

- $G^v$: $G^v(x, y)$ is true if and only if $x > y$.
- $G^v = \{\langle 2, 1 \rangle, \langle 3, 1 \rangle, \langle 3, 2 \rangle\}$
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Satisfiable and Valid

Revisiting the Learning Goals
Evaluating terms and formulas w/o variables

Evaluate these terms and formulas under the valuation $v$.

$f(f(a))$, $(F(a) \lor G(a, b))$.

Valuation $v$:

- **Domain**: $D = \{1, 2, 3\}$.
- **Individuals**: $a^v = 1$, $b^v = 2$, $c^v = 3$.
- **Free variables**: $u^v = 3$, $v^v = 2$, $w^v = 1$.
- **Functions**:
  - $f^v$: $f^v(1) = 2$, $f^v(2) = 3$, $f^v(3) = 1$.
  - $g^v$: $g^v(x, y) = ((x + y) \mod 3) + 1$.
- **Relations**:
  - $F^v$: $F^v(x)$ is true if and only if $x > 5$.
  - $G^v$: $G^v(x, y)$ is true if and only if $x > y$. 
Give a valuation that makes the formula true/false

Complete the valuation $v$ such that

(A) $G(a, f(f(a)))^v = 1$
(B) $G(a, f(f(a)))^v = 0$

Valuation $v$:

- Domain: $D = \{1, 2, 3\}$.
- Individuals: $a^v = \?, b^v = \?, c^v = \?$. 
- Functions: $f^v : \?, g^v : \?$. 
- Relations: $P^v : \?, G^v : \?$
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A valuation for interpreting free variables

Valuation $v$:

- **Domain**: $D = \{1, 2, 3\}$.
- **Individuals**: $a^v = 1$, $b^v = 2$, $c^v = 3$.
- **Free variables**: $u^v = 3$, $v^v = 2$, $w^v = 1$.
- **Functions**:
  
  $f^v$: $f^v(1) = 2$, $f^v(2) = 3$, $f^v(3) = 1$.
  
  $g^v$: $g^v(x, y) = ((x + y) \mod 3) + 1$.

- **Relations**:
  
  $F^v$: $F^v(x)$ is true if and only if $x > 5$.
  
  $G^v$: $G^v(x, y)$ is true if and only if $x > y$. 
Evaluating terms & formulas w/o bound variables

Evaluate these terms and formulas under the valuation $v$.
$g(u, f(b))$, $G(a, f(f(u)))$.

Valuation $v$:

- **Domain:** $D = \{1, 2, 3\}$.
- **Individuals:** $a^v = 1$, $b^v = 2$, $c^v = 3$.
- **Free variables:** $u^v = 3$, $v^v = 2$, $w^v = 1$.
- **Functions:**
  - $f^v$: $f^v(1) = 2$, $f^v(2) = 3$, $f^v(3) = 1$.
  - $g^v$: $g^v(x, y) = ((x + y) \mod 3) + 1$.
- **Relations:**
  - $F^v$: $F^v(x)$ is true if and only if $x > 5$.
  - $G^v$: $G^v(x, y)$ is true if and only if $x > y$. 
Give a valuation that makes the formula true/false

Complete a valuation such that

(A) \( G(a, f(f(u)))^v = 1 \)
(B) \( G(a, f(f(u)))^v = 0 \)

Valuation \( v \):

- Domain: \( D = \{1, 2, 3\} \).
- Individuals: \( a^v = ?, b^v = ?, c^v = ? \).
- Free variables: \( u^v = ?, u^v = ? u^v = ? \).
- Functions: \( f^v : ?, g^v : ? \)
- Relations: \( P^v : ?, G^v : ? \)
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Revisiting the Learning Goals
Evaluate quantified formulas under a valuation

Evaluate these formulas under the valuation $v$.

(A) \((\forall x \ (\exists y \ G(x, y)))\)

(B) \((\exists x \ (\forall y \ G(x, y)))\)

Valuation $v$:

- **Domain:** $D = \{1, 2, 3\}$.
- **Relations:** $G^v = \{\langle 1, 2 \rangle, \langle 3, 1 \rangle, \langle 2, 3 \rangle\}$. 
Complete the valuation $v$ to make the following formula true/false. 
(When satisfying the formula, try making $G^v$ as small as possible.)

(A) $(\forall y \ (\exists x \ G(x,y)))$
(B) $(\exists y \ (\forall x \ G(x,y)))$

Valuation $v$:
- Domain: $D = \{1, 2, 3\}$. 
- ...
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Difference between Individual and Free Variable Symbols

Let our domain be the set of people. Let the predicate $L(u)$ be true if $u$ likes chocolates. Let $a$ be an individual symbol referring to Alice.

- $L(a)$
  This formula only contains individual symbols. Since $a$ refers to Alice, the truth value of this formula is already determined (It’s true because Alice likes chocolates).

- $L(u)$
  This formula only contains free variable symbols. We do not know the truth value of this formula because $u$ can refer to any person in the domain. We need to assign $u$ to a particular person because we can determine whether this formula is true or false.
Difference between Free and Bound Variables

Let our domain be the set of integers.

- \( u + u = v \)
  The variables are free.
  We do not know the truth value of this formula until we assign the free variables to elements of the domain.

- \( \forall x \forall y (x + x = y) \)
  The variables are bound.
  We know the truth value of this formula because the meanings of the variables are given by the quantifiers.
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Revisiting the Learning Goals
A formula $A$ is \textbf{satisfiable}:

there exists a valuation $v$, $A^v = 1$.

A formula $A$ is \textbf{valid}:

for every valuation $v$, $A^v = 1$.

Most predicate formulas are satisfiable but not valid because we have a great deal of freedom to choose the valuation.
Is the following formula satisfiable? If it’s satisfiable, give a valuation that satisfies it. If it’s not satisfiable, give a proof.

$$(\exists x \ F(x)) \rightarrow (\forall x \ F(x))$$
Proving that a formula is valid/not valid

Is the following formula valid? If it’s valid, give a proof. If it’s not valid, give a counterexample.

\[(\forall x \ F(x)) \rightarrow (\exists x \ F(x))\]

- Determine whether the formula is valid or not.

- How do I prove that a formula is NOT valid?

- How do I prove that a formula is valid?
Proving that a formula is valid or not

Is the following formula valid? If it’s valid, give a proof. If it’s not valid, give a counterexample.

\((\forall x \; F(x)) \rightarrow (\exists x \; F(x))\)
Proving that a formula is valid or not

Is the following formula valid? If it’s valid, give a proof. If it’s not valid, give a counterexample.

\((\exists x \ F(x)) \rightarrow (\forall x \ F(x))\)
Revisiting the learning goals

By the end of this lecture, you should be able to:

▶ Define a valuation.
▶ Determine the value of a term given a valuation.
▶ Determine the truth value of a formula given a valuation.
▶ Give a valuation that makes a formula true or false.
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