Predicate Logic: Introduction and Translations

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Lecture 11

Outline

Learning goals

By the end of this lecture, you should be able to

(Introduction to Predicate Logic)

- Give examples of English sentences that can be modeled using predicate logic but cannot be modeled using propositional logic.

(Translations)

- Translate an English sentence into a predicate formula.
- Translate a predicate formula into an English sentence.

Outline

Translate the following argument into propositional logic

Consider the argument below.

1. Every human is mortal.
2. Socrates is a human.

From these two sentences, one should be able to conclude that

3. Socrates is mortal.

In propositional logic, how would you express this argument?

What can we NOT express in propositional logic?

A few things that are difficult to express using propositional logic:

- Relationships among individuals:
  “Alice is a friend of Bob and Alice is not a friend of Cate.“
- Generalizing patterns:
  “Every bear likes honey.“
- Infinite domains:
  “Define what it means for a natural number to be prime.“

We can use predicate logic (first-order logic) to express all of these.

Would you really use predicate logic?

Examples of predicate logic in CS245 so far:

1. Every well-formed formula has an equal number of left and right brackets.
2. If there does not exist a formal deduction proof from the premises to the conclusion, then the premises do not logically imply the conclusion.
3. There exists a truth valuation t such that for every $A \in \Sigma$, $A^t = 1$.

Would you really use predicate logic?

Examples of predicate logic in Computer Science:

1. Data structure: Every key stored in the left subtree of a node N is smaller than the key stored at N.
2. Algorithms: In the worst case, every comparison sort requires at least $cn \log n$ comparisons to sort n values, for some constant $c > 0$.
3. Java example: There is no path via references from any variable in scope to any memory location available for garbage collection...
4. Database query: Select a person whose age is greater than or equal to the age of every other person in the table.

Outline

Elements of predicate logic

Predicate logic generalizes propositional logic.

New things in predicate logic:

- Domains
- Relations
- Quantifiers

Domains

A domain is

- a non-empty set of objects/individuals.
- a world that our statement is situated within.

Examples: natural numbers, people, animals, etc.

Why is it important to specify a domain?

- A statement can have different truth values in different domains.
- There exists a number whose square is 2.
  Is this sentence true when the domain is $\mathbb{N}$? $\mathbb{R}$?

Objects in a domain

Individuals: concrete objects in the domain

- Natural numbers: 0, 6, 100, ...
- Alice, Bob, Eve, ...
- Animals: Winnie the Pooh, Micky Mouse, Simba, ...

Variables: placeholders for concrete objects.

- e.g. u, v, w, x, y, z.
- refers to an object without specifying a particular object.

Relations

A relation represents

- a property of an individual, or
- a relationship among multiple individuals.
- a n-ary function which takes constants and/or variables as inputs and outputs $1/0$

Examples:

- Define $S(x)$ to mean "x is a student". (unary predicate)
    - Bob is a student: $S(Bob)$
    - Micky Mouse is not a student: $(\neg S(\text{Micky Mouse}))$
    - u is a student: $S(u)$
- Define $Y(x, y)$ to mean "x is younger than y". (binary predicate)
    - Alex is younger than Sam: $Y(\text{Alex}, \text{Sam})$
    - u is younger than v: $Y(u, v)$

Quantifiers

For how many objects in the domain is the statement true?

- The universal quantifier $\forall$:
  the statement is true for every object in the domain.
- The existential quantifier $\exists$:
  the statement is true for one or more objects in the domain.

Outline

CQ Translating English into Predicate Logic

Let the domain be the set of animals.
$H(x)$ means that x likes honey. $B(x)$ means that x is a bear.

Are the following translations correct?

1. At least one animal likes honey. $(\exists x\ H(x))$.
2. Not every animal likes honey. $(\neg(\forall x\ H(x)))$.

(A) Both are correct.
(B) 1 is correct and 2 is wrong.
(C) 1 is wrong and 2 is correct.
(D) Both are wrong.

Translating English into Predicate Logic

Let the domain be the set of animals.
$H(x)$ means that x likes honey. $B(x)$ means that x is a bear.

Translate the following sentences into predicate logic.

1. All animals like honey.
2. At least one animal likes honey.
3. Not every animal likes honey.
4. No animal likes honey.

Translating English into Predicate Logic

Let the domain be the set of animals.
$H(x)$ means that x likes honey. $B(x)$ means that x is a bear.

Translate the following sentences into predicate logic.

5. No animal dislikes honey.
6. Not every animal dislikes honey.
7. Some animal dislikes honey.
8. Every animal dislikes honey.

CQ Translating English into Predicate Logic

Consider this sentence: Every bear likes honey.

Which one is the correct translation into predicate logic?

(A) $\forall x \ (B(x) \wedge H(x))$

(B) $\forall x \ (B(x) \vee H(x))$

(C) $\forall x \ (B(x) \rightarrow H(x))$

(D) $\forall x \ (H(x) \rightarrow B(x))$

Let the domain be the set of animals. $H(x)$ means that x likes honey. $B(x)$ means that x is a bear.

CQ Translating English into Predicate Logic

Consider this sentence: Some bear likes honey.
(At least one bear likes honey.)

Which one is the correct translation into predicate logic?

(A) $\exists x \ (B(x) \wedge H(x))$

(B) $\exists x \ (B(x) \vee H(x))$

(C) $\exists x \ (B(x) \rightarrow H(x))$

(D) $\exists x \ (H(x) \rightarrow B(x))$

Let the domain be the set of animals. $H(x)$ means that x likes honey. $B(x)$ means that x is a bear.

Multiple Quantifiers

Let the domain be the set of people. Let $L(x, y)$ mean that person x likes person y.

Translate the following formulas into English.

1. $\forall x \ (\forall y \ L(x, y))$
2. $\exists x \ (\exists y \ L(x, y))$
3. $\forall x \ (\exists y \ L(x, y))$
4. $\exists y \ (\forall x \ L(x, y))$

CQ Translations

Translate the sentence "every student has taken some course." into a predicate formula.

Let x refer to a student and let y refer to a course.
What quantifiers should we use for x and y?

(A) $\forall x$ and $\forall y$

(B) $\forall x$ and $\exists y$

(C) $\exists x$ and $\forall y$

(D) $\exists x$ and $\exists y$

Let the domain contain the set of all students and courses.
Define the following predicates: $S(x)$: x is a student.
$C(y)$: y is a course. $T(x, y)$: student x has taken course y.

CQ Translations

Translate the sentence "every student has taken some course." into a predicate formula.

Which of the following is a correct translation?

(A) $\forall x \; (S(x) \to (\exists y \; C(y) \to T(x, y)))$

(B) $\forall x \; (S(x) \to (\exists y \; C(y) \land T(x, y)))$

(C) $\forall x \; (S(x) \land (\exists y \; C(y) \to T(x, y)))$

(D) $\forall x \; (S(x) \land (\exists y \; C(y) \land T(x, y)))$

Let the domain contain the set of all students and courses.
Define the following predicates: $S(x)$: x is a student.
$C(y)$: y is a course. $T(x, y)$: student x has taken course y.

CQ Translations

Translate the sentence "some student has not taken any course." into a predicate formula.

Let x refer to a student and let y refer to a course.
What quantifiers should we use for x and y?

(A) $\forall x$ and $\forall y$

(B) $\forall x$ and $\exists y$

(C) $\exists x$ and $\forall y$

(D) $\exists x$ and $\exists y$

Let the domain contain the set of all students and courses.
Define the following predicates: $S(x)$: x is a student.
$C(y)$: y is a course. $T(x, y)$: student x has taken course y.

CQ Translations

Translate the sentence "some student has not taken any course." into a predicate formula.

Which of the following is a correct translation?

(A) $\exists x \ (S(x) \rightarrow (\forall y \ C(y) \rightarrow \neg T(x,y)))$

(B) $\exists x \ (S(x) \rightarrow (\forall y \ C(y) \wedge (\neg T(x,y)))$

(C) $\exists x \ (S(x) \wedge (\forall y \ C(y) \rightarrow \neg T(x,y)))$

(D) $\exists x \ (S(x) \wedge (\forall y \ C(y) \wedge \neg T(x,y)))$

Let the domain contain the set of all students and courses.
Define the following predicates: $S(x)$: x is a student.
$C(y)$: y is a course. $T(x,y)$: student x has taken course y.

Outline

CQ Interpreting the quantifiers

Let the domain be $\{Alice, Bob, Eve\}$.
Let $C(x)$ mean that person x likes chocolates.

Which of the following is equivalent to $\forall x\ C(x)$?

(A) $((C(Alice) \wedge C(Bob)) \wedge C(Eve))$

(B) $((C(Alice) \vee C(Bob)) \vee C(Eve))$

(C) $((C(Alice) \rightarrow C(Bob)) \rightarrow C(Eve))$

(D) $((C(Alice) \leftrightarrow C(Bob)) \leftrightarrow C(Eve))$

(E) None of the above

CQ Interpreting the quantifiers

Let the domain be $\{Alice, Bob, Eve\}$.
Let $C(x)$ mean that person x likes chocolates.

Which of the following is equivalent to $\exists x\ C(x)$?

(A) $((C(Alice) \land C(Bob)) \land C(Eve))$

(B) $((C(Alice) \lor C(Bob)) \lor C(Eve))$

(C) $((C(Alice) \rightarrow C(Bob)) \rightarrow C(Eve))$

(D) $((C(Alice) \leftrightarrow C(Bob)) \leftrightarrow C(Eve))$

(E) None of the above

Interpreting the quantifiers

Let $D = \{d_1, d_2, ..., d_n\}$.

$\forall$ is a big AND:

$\forall x\ P(x)$ is equivalent to ...$(P(d_1) \wedge P(d_2)) \wedge ... \wedge P(d_n)$.

$\exists$ is a big OR:

$\exists x\ P(x)$ is equivalent to ...$(P(d_1) \vee P(d_2)) \vee ... \vee P(d_n)$.

Negating a quantified formula

"Generalized De Morgan's Laws"

- $\neg(\forall x \; P(x)) \models\!\!\mid \exists x \; (\neg P(x))$
- $\neg(\exists x \; P(x)) \models\!\!\mid \forall x \; (\neg P(x))$

Outline

Why do we need functions?

Translate the sentence. "every child is younger than their mother."

1. The domain is the set of people.
   $Child(x)$ means x is a child.
   $IsMother(x, y)$ means x is y's mother.
   $IsYounger(x, y)$ means x is younger than y.
2. Use the same set of definitions above
   and the function $mother(x)$ which returns x's mother.

Why do we need functions?

Using functions allows us
to avoid ugly/inelegant predicate formulas.

Try translating the following sentence with and without functions.
"Andy and Paul have the same maternal grandmother."

A Summary of Translation Idioms

- Every/All/Each/Any
  $\forall x$
- Some/At least one/There exists a/There is a
  $\exists x$
- None/No x
  $\neg(\exists x \dots)$
- Not every/Not all
  $\neg(\forall x \dots)$
- Every P-ish x has property Q
  $\forall x \; P(x) \rightarrow Q(x)$
- Some P-ish x has property Q
  $\exists x \; P(x) \wedge Q(x)$

Outline

At least, at most, and exactly

Let the domain be the set of animals. Let $B(x)$ be that x is a bear.

1. There are at least two bears.
2. There are at most one bear.
3. There are exactly one bear.

Revisiting the learning goals

By the end of this lecture, you should be able to

(Introduction to Predicate Logic)

- Give examples of English sentences that can be modeled using predicate logic but cannot be modeled using propositional logic.

(Translations)

- Translate an English sentence into a predicate formula.
- Translate a predicate formula into an English sentence.