Propositional Logic: Soundness of Formal Deduction

Alice Gao

Lecture 9
Learning Goals

By the end of this lecture, you should be able to

▶ Define the soundness of formal deduction.
▶ Prove that a tautological consequence holds using formal deduction and the soundness of formal deduction.
▶ Show that no formal deduction proof exists using the contraposition of the soundness of formal deduction.
Tautological Consequence

Let $\Sigma$ be a set of propositional formulas. Let $A$ be a propositional formula.

$$\Sigma \models A$$

- $\Sigma$ semantically implies $A$.
- $A$ is a tautological consequence of $\Sigma$.
- For any truth valuation $t$, if every formula in $\Sigma$ is true under $t$ ($\Sigma^t = 1$), then $A$ is also true under $t$ ($A^t = 1$).

Several ways of proving a tautological consequence: truth table, direct proof, a proof by contradiction, etc.
Formal Deduction

Let $\Sigma$ be a set of propositional formulas. Let $A$ be a propositional formula.

\[ \Sigma \vdash A \]

- $\Sigma$ formally proves $A$.
- There exists a proof which syntactically transforms the premises in $\Sigma$ to produce the conclusion $A$.
- A formal proof is a syntactic manipulation of symbols and it can be checked mechanically.
Tautological Consequence v.s. Formal Deduction

\[ \Sigma \models A \] and \[ \Sigma \vdash A \] appear to be similar. Ideally, we would like them to be equivalent. This could mean two properties:

1. If \( \Sigma \vdash A \), then \( \Sigma \models A \). (Soundness of formal deduction)
   If there exists a formal proof from \( \Sigma \) to \( A \), then \( \Sigma \) tautologically implies \( A \).
   (Everything I can formally prove is a tautological consequence.)

2. If \( \Sigma \models A \), then \( \Sigma \vdash A \). (Completeness of formal deduction)
   If \( \Sigma \) tautologically implies \( A \), there exists a formal proof from \( \Sigma \) to \( A \).
   (I can formally prove every tautological consequence.)
Soundness and Completeness of Formal Deduction

Theorem: Formal Deduction is both sound and complete.

Soundness of Formal Deduction means that the conclusion of a proof is always a logical consequence of the premises. That is,

\[ \text{If } \Sigma \models \alpha, \text{ then } \Sigma \vdash \alpha \]

Completeness of Formal Deduction means that all logical consequences in propositional logic are provable in Formal Deduction. That is,

\[ \text{If } \Sigma \vdash \alpha, \text{ then } \Sigma \models \alpha \]
Other proof systems

- resolution
- axiomatic systems
- semantic tableaux
- intuitionistic logic: sound but not complete. e.g. it cannot prove $p \lor (\neg p)$
- any system plus $p \land (\neg p)$ as an axiom: not sound but complete.
  not sound because we can prove $p \land (\neg p)$ which is false. complete because we can prove anything with $p \land (\neg p)$ as an axiom.
Proving the soundness of formal deduction

We will prove this by structural induction on the proof for $\Sigma \vdash A$.

A proof is a recursive structure.

A proof either
- derives the conclusion without using any inference rule, or
  (Base case)
- derives the conclusion by applying a rule of formal deduction on a proof. (Inductive case)
Proof of the soundness of formal deduction

Theorem: For a set of propositional formulas $\Sigma$ and a propositional formula $A$, if $\Sigma \vdash A$, then $\Sigma \models A$.

Proof: We prove this by structural induction on the proof for $\Sigma \vdash A$.

Base case: Assume that there is a proof for $\Sigma \vdash A$ where $A \in \Sigma$. Consider a truth valuation such that $\Sigma^t = 1$. Since $A \in \Sigma$, then $A^t = 1$. Thus, $\Sigma \models A$.

(To be continued)
Proof of the soundness of formal deduction

Induction step: Consider several cases for the last rule applied in the proof of $\Sigma \vdash A$. (There is one case for every rule of formal deduction.)

▶ Assume that the proof of $\Sigma \vdash A$ applies the rule $\land^+$ with the two premises $\Sigma \vdash B$ and $\Sigma \vdash C$ and reaches the conclusion $\Sigma \vdash B \land C$.

Let me prove this case for you.

Induction hypotheses: Assume that $\Sigma \models B$ and $\Sigma \models C$. We need to prove that $\Sigma \models B \land C$.

Consider a truth valuation $t$ such that $\Sigma^t = 1$. By the induction hypotheses, $B^t = 1$ and $C^t = 1$. By the truth table of $\land$, $(B \land C)^t = 1$. Therefore, $\Sigma \models (B \land C)$.

(To be continued)
Proof of the soundness of formal deduction

Induction step (continued):

Assume that the proof of $\Sigma \vdash A$ applies the rule $\rightarrow -$ with the two premises $\Sigma \vdash B$ and $\Sigma \vdash (B \rightarrow C)$ and reaches the conclusion $\Sigma \vdash C$.

Try proving this case yourself.
Applications of soundness and completeness

1. The following inference rule is called Disjunctive syllogism.

   if $\Sigma \vdash \neg A$ and $\Sigma \vdash A \lor B$, then $\Sigma \vdash B$.

   where $A$ and $B$ are well-formed propositional formulas.

   Prove that this inference rule is sound.
   That is, prove that if $\Sigma \models \neg A$ and $\Sigma \models A \lor B$, then $\Sigma \models B$.

2. Show that there does not exist a formal deduction proof for
   $p \lor q \vdash p$, where $p$ and $q$ are propositional variables.

3. Prove that $(A \rightarrow B) \not\vdash (B \rightarrow A)$ where $A$ and $B$ are
   propositional formulas.
Applications of soundness and completeness

The following inference rule is called Disjunctive syllogism.

\[
\text{if } \Sigma \vdash \neg A \text{ and } \Sigma \vdash A \lor B, \text{ then } \Sigma \vdash B.
\]

where \( A \) and \( B \) are well-formed propositional formulas.

Prove that this inference rule is sound.
That is, prove that if \( \Sigma \models \neg A \) and \( \Sigma \models A \lor B \), then \( \Sigma \models B \).

Proof:
Consider a truth valuation \( t \) under which \( \Sigma^t = 1 \). Since \( \Sigma \models (\neg A) \) and \( \Sigma \models A \lor B \), we have that \((\neg A)^t = 1\) and \((A \lor B)^t = 1\). We need to show that \( B^t = 1 \).
By the truth table of \( \neg \), since \((\neg A)^t = 1\), \( A^t = 0 \).
By the truth table of \( \lor \), since \((A \lor B)^t = 1\), at least one of \( A \) and \( B \) is true under \( t \). Since \( A^t = 0 \), then \( B^t = 1 \).

Therefore, \( \Sigma \models B \) holds. QED
Show that there does not exist a formal proof for \( p \lor q \vdash p \), where \( p \) and \( q \) are propositional variables.

**Proof:**

By the contrapositive of the soundness of formal deduction, if \( p \lor q \not\models p \), then \( p \lor q \not\vdash p \). Consider the truth valuation \( t \) where \( p^t = 0 \) and \( q^t = 1 \). By the truth table of \( \lor \), \((p \lor q)^t = 1\). Thus, \( p \lor q \not\models p \). Therefore, \( p \lor q \not\vdash p \).

QED
Prove that \((A \rightarrow B) \not\models (B \rightarrow A)\) where \(A\) and \(B\) are propositional formulas.

Proof:

By the contrapositive of the soundness of formal deduction, if \((A \rightarrow B) \not\models (B \rightarrow A)\), then \((A \rightarrow B) \not\models (B \rightarrow A)\). We need to give a counterexample to show that \((A \rightarrow B) \not\models (B \rightarrow A)\).

Let \(A = p\) and \(B = q\). Consider the truth valuation where \(p^t = 0\) and \(q^t = 1\). By the truth table of \(\rightarrow\), \((p \rightarrow q)^t = 1\) and \((q \rightarrow p)^t = 0\). Therefore, \((A \rightarrow B) \not\models (B \rightarrow A)\) and \((A \rightarrow B) \not\models (B \rightarrow A)\).

QED
Revisiting the Learning Goals

By the end of this lecture, you should be able to

▶ Define the soundness of formal deduction.
▶ Prove that a tautological consequence holds using formal deduction and the soundness of formal deduction.
▶ Show that no formal deduction proof exists using the contrapositive of the soundness of formal deduction.