Propositional Logic: Soundness of Formal Deduction

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Lecture 9
Learning Goals

By the end of this lecture, you should be able to

▶ Define the soundness of formal deduction.
▶ Prove that a tautological consequence holds using formal deduction and the soundness of formal deduction.
▶ Show that no formal deduction proof exists using the contrapositive of the soundness of formal deduction.
Tautological Consequence

Let $\Sigma$ be a set of propositional formulas. Let $A$ be a propositional formula.

$$\Sigma \vdash A$$

- $\Sigma$ semantically implies $A$.
- $A$ is a tautological consequence of $\Sigma$.
- For any truth valuation $t$, if every formula in $\Sigma$ is true under $t$ ($\Sigma^t = 1$), then $A$ is also true under $t$ ($A^t = 1$).

Several ways of proving a tautological consequence: truth table, direct proof, a proof by contradiction, etc.
Let $\Sigma$ be a set of propositional formulas. Let $A$ be a propositional formula.

$$\Sigma \vdash A$$

- $\Sigma$ formally proves $A$.
- There exists a proof which syntactically transforms the premises in $\Sigma$ to produce the conclusion $A$.
- A formal proof is a syntactic manipulation of symbols and it can be checked mechanically.
Tautological Consequence v.s. Formal Deduction

Σ ⊨ A and Σ ⊢ A appear to be similar. Ideally, we would like them to be equivalent. This could mean two properties:

1. If Σ ⊢ A, then Σ ⊨ A. (Soundness of formal deduction)
   If there exists a formal proof from Σ to A, then Σ tautologically implies A.
   (Everything I can formally prove is a tautological consequence.)

2. If Σ ⊨ A, then Σ ⊢ A. (Completeness of formal deduction)
   If Σ tautologically implies A, there exists a formal proof from Σ to A.
   (I can formally prove every tautological consequence.)
Soundness and Completeness of Formal Deduction

Theorem: Formal Deduction is both sound and complete.

Soundness of Formal Deduction means that the conclusion of a proof is always a logical consequence of the premises. That is,

\[ \text{If } \Sigma \vdash \alpha, \text{ then } \Sigma \models \alpha \]

Completeness of Formal Deduction means that all logical consequences in propositional logic are provable in Formal Deduction. That is,

\[ \text{If } \Sigma \models \alpha, \text{ then } \Sigma \vdash \alpha \]
Other proof systems

- resolution
- axiomatic systems
- semantic tableaux
- intuitionistic logic: sound but not complete. e.g. it cannot prove $p \lor \lnot p$
- any system plus $p \land \lnot p$ as an axiom: not sound but complete.
  not sound because we can prove $p \land \lnot p$ which is false.
  complete because we can prove anything with $p \land \lnot p$ as an axiom.
Proving the soundness of formal deduction

We will prove this by structural induction on the proof for $\Sigma \vdash A$.

A proof is a recursive structure.

A proof either

- derives the conclusion without using any inference rule, or
  (Base case)
- derives the conclusion by applying a rule of formal deduction on a proof. (Inductive case)
Theorem: For a set of propositional formulas $\Sigma$ and a propositional formula $A$, if $\Sigma \vdash A$, then $\Sigma \models A$.

Proof: We prove this by structural induction on the proof for $\Sigma \vdash A$.

Base case: Assume that there is a proof for $\Sigma \vdash A$ where $A \in \Sigma$. Consider a truth valuation such that $\Sigma^t = 1$. Since $A \in \Sigma$, then $A^t = 1$. Thus, $\Sigma \models A$.

(To be continued)
Proof of the soundness of formal deduction

Induction step: Consider several cases for the last rule applied in the proof of $\Sigma \vdash A$. (There is one case for every rule of formal deduction.)

- Assume that the proof of $\Sigma \vdash A$ applies the rule $\land+$ with the two premises $\Sigma \vdash B$ and $\Sigma \vdash C$ and reaches the conclusion $\Sigma \vdash B \land C$.

Let me prove this case for you.

Induction hypotheses: Assume that $\Sigma \models B$ and $\Sigma \models C$. We need to prove that $\Sigma \models B \land C$.

Consider a truth valuation $t$ such that $\Sigma^t = 1$. By the induction hypotheses, $B^t = 1$ and $C^t = 1$. By the truth table of $\land$, $(B \land C)^t = 1$. Therefore, $\Sigma \models (B \land C)$.

(To be continued)
Induction step (continued):

Assume that the proof of $\Sigma \vdash A$ applies the rule $\rightarrow -$ with the two premises $\Sigma \vdash B$ and $\Sigma \vdash (B \rightarrow C)$ and reaches the conclusion $\Sigma \vdash C$.

Try proving this case yourself.
Applications of soundness and completeness

1. The following inference rule is called Disjunctive syllogism.

   \[
   \text{if } \Sigma \vdash \neg A \text{ and } \Sigma \vdash A \lor B, \text{ then } \Sigma \vdash B.
   \]

   where \(A\) and \(B\) are well-formed propositional formulas.

   Prove that this inference rule is sound.

   That is, prove that if \(\Sigma \models \neg A\) and \(\Sigma \models A \lor B\), then \(\Sigma \models B\).

2. Show that there does not exist a formal deduction proof for
   \(p \lor q \vdash p\), where \(p\) and \(q\) are propositional variables.

3. Prove that \((A \rightarrow B) \not\models (B \rightarrow A)\) where \(A\) and \(B\) are propositional formulas.
Applications of soundness and completeness

The following inference rule is called Disjunctive syllogism.

if $\Sigma \vdash \neg A$ and $\Sigma \vdash A \lor B$, then $\Sigma \vdash B$.

where $A$ and $B$ are well-formed propositional formulas.

Prove that this inference rule is sound.
That is, prove that if $\Sigma \models \neg A$ and $\Sigma \models A \lor B$, then $\Sigma \models B$.

Proof:
Consider a truth valuation $t$ under which $\Sigma^t = 1$. Since $\Sigma \models (\neg A)$ and $\Sigma \models A \lor B$, we have that $(\neg A)^t = 1$ and $(A \lor B)^t = 1$. We need to show that $B^t = 1$.

By the truth table of $\neg$, since $(\neg A)^t = 1$, $A^t = 0$.

By the truth table of $\lor$, since $(A \lor B)^t = 1$, at least one of $A$ and $B$ is true under $t$. Since $A^t = 0$, then $B^t = 1$.

Therefore, $\Sigma \models B$ holds. QED
Applications of soundness and completeness

Show that there does not exist a formal proof for $p \lor q \vdash p$, where $p$ and $q$ are propositional variables.

Proof:
By the contrapositive of the soundness of formal deduction, if $p \lor q \not\models p$, then $p \lor q \not\models p$. Consider the truth valuation $t$ where $p^t = 0$ and $q^t = 1$. By the truth table of $\lor$, $(p \lor q)^t = 1$. Thus, $p \lor q \not\models p$. Therefore, $p \lor q \not\models p$.

QED
Applications of soundness and completeness

Prove that \((A \rightarrow B) \not\vdash (B \rightarrow A)\)
where \(A\) and \(B\) are propositional formulas.

Proof:

By the contrapositive of the soundness of formal deduction, if
\((A \rightarrow B) \not\vDash (B \rightarrow A)\), then \((A \rightarrow B) \not\vDash (B \rightarrow A)\). We need to
give a counterexample to show that \((A \rightarrow B) \not\vDash (B \rightarrow A)\).

Let \(A = p\) and \(B = q\). Consider the truth valuation where \(p^t = 0\)
and \(q^t = 1\). By the truth table of \(\rightarrow\), \((p \rightarrow q)^t = 1\) and
\((q \rightarrow p)^t = 0\). Therefore, \((A \rightarrow B) \not\vDash (B \rightarrow A)\) and
\((A \rightarrow B) \not\vDash (B \rightarrow A)\).

QED
Revisiting the Learning Goals

By the end of this lecture, you should be able to

- Define the soundness of formal deduction.
- Prove that a tautological consequence holds using formal deduction and the soundness of formal deduction.
- Show that no formal deduction proof exists using the contrapositive of the soundness of formal deduction.