Propositional Logic:
Soundness of Formal Deduction

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Lecture 9
Learning Goals

By the end of this lecture, you should be able to

▶ Define the soundness of formal deduction.
▶ Prove that a tautological consequence holds using formal deduction and the soundness of formal deduction.
▶ Show that no formal deduction proof exists using the contrapositive of the soundness of formal deduction.
Tautological Consequence

Let $\Sigma$ be a set of propositional formulas. Let $A$ be a propositional formula.

$$\Sigma \models A$$

- $\Sigma$ semantically implies $A$.
- $A$ is a tautological consequence of $\Sigma$.
- For any truth valuation $t$, if every formula in $\Sigma$ is true under $t$ ($\Sigma^t = 1$), then $A$ is also true under $t$ ($A^t = 1$).

Several ways of proving a tautological consequence: truth table, direct proof, a proof by contradiction, etc.
Formal Deduction

Let $\Sigma$ be a set of propositional formulas. Let $A$ be a propositional formula.

$\Sigma \vdash A$

- $\Sigma$ formally proves $A$.
- There exists a proof which syntactically transforms the premises in $\Sigma$ to produce the conclusion $A$.
- A formal proof is a syntactic manipulation of symbols and it can be checked mechanically.
Tautological Consequence v.s. Formal Deduction

Σ ⊨ A and Σ ⊢ A appear to be similar. Ideally, we would like them to be equivalent. This could mean two properties:

1. If Σ ⊢ A, then Σ ⊨ A. (Soundness of formal deduction) If there exists a formal proof from Σ to A, then Σ tautologically implies A.

2. If Σ ⊨ A, then Σ ⊢ A. (Completeness of formal deduction) If Σ tautologically implies A, there exists a formal proof from Σ to A.
Soundness and Completeness of Formal Deduction

Theorem: Formal Deduction is both sound and complete.

Soundness of Formal Deduction means that the conclusion of a proof is always a logical consequence of the premises. That is,

\[ \text{If } \Sigma \vdash \alpha, \text{ then } \Sigma \vDash \alpha \]

Completeness of Formal Deduction means that all logical consequences in propositional logic are provable in Formal Deduction. That is,

\[ \text{If } \Sigma \vDash \alpha, \text{ then } \Sigma \vdash \alpha \]
Other proof systems

- resolution
- axiomatic systems
- semantic tableaux
- intuitionistic logic: sound but not complete. e.g. it cannot prove $p \lor (\neg p)$
- any system plus $p \land (\neg p)$ as an axiom: not sound but complete.
  not sound because we can prove $p \land (\neg p)$ which is false.
  complete because we can prove anything with $p \land (\neg p)$ as an axiom.
We will prove this by structural induction on the proof for $\Sigma \vdash A$.

A proof is a recursive structure.

A proof either

1. derives the conclusion without using any inference rule, or
   (Base case)
2. derives the conclusion by applying a rule of formal deduction on a proof. (Inductive case)
Proof of the soundness of formal deduction

Theorem: For a set of propositional formulas $\Sigma$ and a propositional formula $A$, if $\Sigma \vdash A$, then $\Sigma \models A$.

Proof: We prove this by structural induction on the proof for $\Sigma \vdash A$.

Base case: Assume that there is a proof for $\Sigma \vdash A$ where $A \in \Sigma$. Consider a truth valuation such that $\Sigma^t = 1$. Since $A \in \Sigma$, then $A^t = 1$. Thus, $\Sigma \models A$.

(To be continued)
Proof of the soundness of formal deduction

Induction step: Consider several cases for the last rule applied in the proof of $\Sigma \vdash A$. (There is one case for every rule of formal deduction.)

- Assume that the proof of $\Sigma \vdash A$ applies the rule $\land+$ with the two premises $\Sigma \vdash B$ and $\Sigma \vdash C$ and reaches the conclusion $\Sigma \vdash B \land C$.

Let me prove this case for you.

(To be continued)
Proof of the soundness of formal deduction

Induction step (continued):

Assume that the proof of $\Sigma \vdash A$ applies the rule $\rightarrow -$ with the two premises $\Sigma \vdash B$ and $\Sigma \vdash (B \rightarrow C')$ and reaches the conclusion $\Sigma \vdash C'$.

Try proving this case yourself.
1. The following inference rule is called **Disjunctive syllogism**.

   \[ \text{if } \Sigma \vdash \neg A \text{ and } \Sigma \vdash A \lor B, \text{ then } \Sigma \vdash B. \]

   where \( A \) and \( B \) are well-formed propositional formulas.

   Prove that this inference rule is sound.

   That is, prove that if \( \Sigma \models \neg A \) and \( \Sigma \models A \lor B \), then \( \Sigma \models B \).

2. Show that there does not exist a formal deduction proof for
   \[ p \lor q \vdash p, \]  
   where \( p \) and \( q \) are propositional variables.

3. Prove that \( (A \rightarrow B) \not\vdash (B \rightarrow A) \) where \( A \) and \( B \) are
   propositional formulas.
Applications of soundness and completeness

The following inference rule is called Disjunctive syllogism.

\[
\text{if } \Sigma \vdash \neg A \text{ and } \Sigma \vdash A \lor B, \text{ then } \Sigma \vdash B.
\]

where \( A \) and \( B \) are well-formed propositional formulas.

Prove that this inference rule is sound.
That is, prove that if \( \Sigma \vdash \neg A \) and \( \Sigma \vdash A \lor B \), then \( \Sigma \vdash B \).
Applications of soundness and completeness

Show that there does not exist a formal proof for \( p \lor q \vdash p \), where \( p \) and \( q \) are propositional variables.
Applications of soundness and completeness

Prove that \((A \rightarrow B) \not\models (B \rightarrow A)\) where \(A\) and \(B\) are propositional formulas.

Proof:

By the contrapositive of the soundness of formal deduction, if \((A \rightarrow B) \not\models (B \rightarrow A)\), then \((A \rightarrow B) \not\models (B \rightarrow A)\). We need to give a counterexample to show that \((A \rightarrow B) \not\models (B \rightarrow A)\).

Let \(A = p\) and \(B = q\). Consider the truth valuation where \(p^t = 0\) and \(q^t = 1\). By the truth table of \(\rightarrow\), \((p \rightarrow q)^t = 1\) and \((q \rightarrow p)^t = 0\). Therefore, \((A \rightarrow B) \not\models (B \rightarrow A)\) and \((A \rightarrow B) \not\models (B \rightarrow A)\).

QED
Revisiting the Learning Goals

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