Propositional Logic: Tautological Consequence and Translations

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Lecture 6

Outline

Learning goals

By the end of this lecture, you should be able to

- Determine if a set of formulas is satisfiable.
- Define tautological consequence. Explain subtleties of tautological consequence.
- Prove that a tautological consequence holds/does not hold by using the definition of tautological consequence, and/or truth tables.
- Translate an English sentence with no logical ambiguity into a propositional formula.
- Translate an English sentence with logical ambiguity into multiple propositional formulas and prove that the propositional formulas are not tautologically equivalent.

Logical Deduction and Tautological Consequence

- Logic is the science of reasoning.
- The process of logical deduction is formalized by the notion of tautological consequence.
- Can we deduce a conclusion based on a set of premises?

Outline

Satisfying a Set of Formulas

Let Σ denote any set of formulas.

$$\Sigma^t = \begin{cases} 1, & \text{if for each } B \in \Sigma, B^t = 1, \\ 0, & \text{otherwise} \end{cases}$$

What does $\Sigma^t = 0$ mean?

Definition (Satisfiability)

Σ is satisfiable if and only if there is some truth valuation t such that $\Sigma^t = 1$. When $\Sigma^t = 1$, t is said to satisfy Σ.

CQ Is Sigma Satisfiable?

Outline

Tautological Consequence

Definition (Tautological Consequence)

Suppose $\Sigma \subseteq Form(L^p)$ and $A \in Form(L^p)$.

A is a tautological consequence of Σ (that is, of the formulas in Σ), written as $\Sigma \vDash A$, if and only if
for any truth valuation t, $\Sigma^t = 1$ implies $A^t = 1$.

Tautological Equivalence

A and B are (tautologically) equivalent if and only if $A \models\mathrel{\mkern-9mu}\dashv B$ holds.

$A \models\mathrel{\mkern-9mu}\dashv B$ denotes $A \models B$ and $B \models A$.

Outline

CQ: Prove a tautological consequence

CQ: Consider the tautological consequence $\Sigma \models A$.
To prove that the tautological consequence holds,
we need to consider

(A) Every truth valuation t under which $\Sigma^t = 1$.
(B) Every truth valuation t under which $\Sigma^t = 0$.
(C) One truth valuation t under which $\Sigma^t = 1$.
(D) One truth valuation t under which $\Sigma^t = 0$.

CQ: Disprove a tautological consequence

CQ: Consider the tautological consequence $\Sigma \models A$.
To prove that the tautological consequence does NOT hold, we need to consider

(A) Every truth valuation t under which $\Sigma^t = 1$ and $A^t = 1$.
(B) Every truth valuation t under which $\Sigma^t = 1$ and $A^t = 0$.
(C) One truth valuation t under which $\Sigma^t = 1$ and $A^t = 1$.
(D) One truth valuation t under which $\Sigma^t = 1$ and $A^t = 0$.

CQ: Proving/disproving a tautological consequence using a truth table

CQ: Let $\Sigma = \{\neg(p \wedge q), p \to q\}$, $x = \neg p$, and $y = p \leftrightarrow q$.
Based on the truth table, which of the following statements is true?

A) $\Sigma \models x$ and $\Sigma \models y$.

B) $\Sigma \models x$ and $\Sigma \not\models y$.

C) $\Sigma \not\models x$ and $\Sigma \models y$.

D) $\Sigma \not\models x$ and $\Sigma \not\models y$.

p	q	$(\neg(p \wedge q))$	$(p \to q)$	$(\neg p)$	$(p \leftrightarrow q)$
0	0	1	1	1	1
0	1	1	1	1	0
1	0	1	0	0	0
1	1	0	1	0	1

Prove a tautological consequence using the definition

Exercise. Show that $\{(\neg(p \wedge q)), (p \rightarrow q)\} \vDash (\neg p)$.

Disprove a tautological consequence using the definition

Exercise. Show that $\{(\neg(p \wedge q)), (p \rightarrow q)\} \not\models (p \leftrightarrow q)$.

Disproving propositional logical consequence

A student is trying to prove that $\{(A \to B)\} \not\models (B \to A)$ where A and B are well-formed predicate formulas. The student starts the proof by writing down the following sentence.

> *There exists a truth valuation t such that $B^t = 1$ and $A^t = 0$.*

Is the above sentence true (a valid claim)?

(A) Yes, it is true.

(B) No, it is false.

(C) There is not enough information to tell.

Outline

Subtleties of a Tautological Consequence

Consider the tautological consequence $\Sigma \vDash A$.
Does the tautological consequence hold
under each of the following conditions?

1. Σ is the empty set.
2. Σ is not satisfiable.
3. A is a tautology.
4. A is a contradiction.

CQ Subtleties of a Tautological Consequence

Outline

English sentences with no logical ambiguity

Translate the following sentences to propositional formulas.

1. Nadhi eats a fruit if the fruit is an apple.
   Nadhi eats a fruit only if the fruit is an apple.
2. Soo-Jin will eat an apple or an orange but not both.
3. If it is sunny tomorrow, then I will play golf, provided that I am relaxed.

English sentences with logical ambiguity

Give multiple translations of the following sentences into propositional logic.

1. Sidney will carry an umbrella unless it is sunny.
2. Pigs can fly and the grass is red or the sky is blue.

Translations: A reference page

- $\neg p$: p does not hold; p is false; it is not the case that p
- $p \wedge q$: p but q; not only p but q; p while q; p despite q; p yet q; p although q
- $p \vee q$: p or q or both; p and/or q;
- $p \to q$: p implies q; q if p; p only if q; q when p; p is sufficient for q; q is necessary for p
- $p \leftrightarrow q$: p is equivalent to q; p exactly if q; p is necessary and sufficient for q

Revisiting the learning goals

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