Propositional Logic: Semantics

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Lecture 5
Outline

Learning Goals

Truth valuation

The meanings of connectives

Tautology, Contradiction, Satisfiable

Revisiting the Learning Goals
Learning goals

By the end of this lecture, you should be able to

▶ Define a truth valuation.
▶ Determine the truth value of a formula given a truth valuation.
▶ Give a truth valuation under which a formula is true or false.
▶ Evaluate the truth value of a formula using a truth table and/or a valuation tree.
▶ Determine if a formula is a tautology, a contradiction, or satisfiable but not a tautology.
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The meaning of well-formed formulas

How do we interpret the propositional language $L^p$ and make the formulas express propositions?

Formulas are composed of atoms (proposition symbols) and connectives.

To interpret a formula, we have to give meanings to the atoms and the connectives.
Truth tables of connectives

The unary connective $\neg$:

$$
\begin{array}{c|c}
A & (\neg A) \\
\hline
1 & 0 \\
0 & 1 \\
\end{array}
$$

The binary connectives $\land$, $\lor$, $\rightarrow$, and $\leftrightarrow$:

$$
\begin{array}{c|c|c|c|c|c}
A & B & (A \land B) & (A \lor B) & (A \rightarrow B) & (A \leftrightarrow B) \\
\hline
1 & 1 & 1 & 1 & 1 & 1 \\
1 & 0 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 \\
\end{array}
$$
Definition of a truth valuation

Definition (Truth valuation)

A truth valuation is a function with the set of all proposition symbols as domain and \( \{1, 0\} \) as range.

We use \( t \) to denote any truth valuation.

The value which \( t \) assigns to any formula \( A \) is written as \( A^t \).
Values of formulas

The value assigned to formulas by a truth valuation $t$ is defined by recursion:

$$ p^t \in \{1, 0\}. \quad (\neg A)^t = \begin{cases} 1, & \text{if } A^t = 0, \\ 0, & \text{otherwise}. \end{cases} $$

$$ (A \land B)^t = \begin{cases} 1, & \text{if } A^t = B^t = 1, \\ 0, & \text{otherwise}. \end{cases} \quad (A \lor B)^t = \begin{cases} 1, & \text{if } A^t = 1 \text{ or } B^t = 1, \\ 0, & \text{otherwise}. \end{cases} $$

$$ (A \to B)^t = \begin{cases} 1, & \text{if } A^t = 0 \text{ or } B^t = 1, \\ 0, & \text{otherwise}. \end{cases} \quad (A \leftrightarrow B)^t = \begin{cases} 1, & \text{if } A^t = B^t, \\ 0, & \text{otherwise}. \end{cases} $$
What is a truth valuation, intuitively?

Have you watched the TV series “Fringe”?

A truth valuation defines a parallel universe.

Our universe: The sky is blue and pigs do not fly.
Parallel universe 1: The sky is red, and pigs do not fly.
Parallel universe 2: The sky is blue, and pigs fly.
A structural induction example

Theorem: For any $A \in \text{Form}(L^p)$ and any truth valuation $t$, $A^t \in \{1, 0\}$.

Proof.
By induction on the structure of $A$. 

CQ Evaluating the truth value of a formula
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Revisiting the Learning Goals
### Or, Exclusive Or, and Equivalence

<table>
<thead>
<tr>
<th>$A$</th>
<th>$B$</th>
<th>$A \lor B$</th>
<th>$((A \land (\neg B)) \lor ((\neg A) \land B))$</th>
<th>$A \leftrightarrow B$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
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- Difference between the “or” and the exclusive or?
- Relationship between the exclusive or and the equivalence?
CQ Understanding an implication
Proving/disproving an implication

Consider the following implications.

(A) “If Alexandra is rich, she pays your tuition.”

(B) \(((p \land q) \rightarrow (p \lor q))\).

1. How do you prove that the implication is false?

2. How do you prove that the implication is true?
Think of an implication as a promise that someone made to you. In what case can you prove that the promise has been broken (i.e. the implication is false)?

- When the premise is true, what is the relationship between the truth value of the conclusion and the truth value of the implication?

- When the premise is false, the implication is vacuously true. Could you come up with an intuitive explanation for this?

- If the conclusion is true, is the implication true or false?

- The implication \((a \rightarrow b)\) is logically equivalent to \(((\neg a) \vee b)\). Does this equivalent formula make sense to you? Explain.
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Revisiting the Learning Goals
A formula $A$ is a **tautology** if and only if

For any truth valuation $t$, $A^t = 1$.

A formula $A$ is a **contradiction** if and only if

For any truth valuation $t$, $A^t = 0$.

A formula $A$ is **satisfiable** if and only if

There exists a truth valuation $t$ such that $A^t = 1$.  

Is a formula a tautology, contradiction, and/or satisfiable?

How do we determine if a formula is a tautology, a contradiction, and/or satisfiable?

Three approaches:

- Draw a truth table.
- Draw a valuation tree (a more compact truth table).
- Use reasoning to get an answer quickly.
Properties and truth tables

- **Tautology**: Each formula is true in EVERY/AT LEAST ONE/NO row of its truth table false in EVERY/AT LEAST ONE/NO row of its truth table.

- **Satisfiable but not a tautology**: Each formula is true in EVERY/AT LEAST ONE/NO row of its truth table false in EVERY/AT LEAST ONE/NO row of its truth table.

- **Contradiction**: Each formula is true in EVERY/AT LEAST ONE/NO row of its truth table false in EVERY/AT LEAST ONE/NO row of its truth table.
Simplifying a formula

After plugging in the value of a variable, we can simplify a formula as follows:

\[
\begin{align*}
p \land 1 & \equiv \\
p \land 0 & \equiv \\
p \land p & \equiv \\
p \lor 1 & \equiv \\
p \lor 0 & \equiv \\
p \lor p & \equiv \\
p \rightarrow 1 & \equiv \\
p \rightarrow 0 & \equiv \\
1 \rightarrow p & \equiv \\
0 \rightarrow p & \equiv \\
p \rightarrow p & \equiv \\
p \leftrightarrow 1 & \equiv \\
p \leftrightarrow 0 & \equiv \\
p \leftrightarrow p & \equiv
\end{align*}
\]

We can evaluate a formula by constructing a valuation tree using these rules.
A valuation tree

Prove that the following formula is a tautology using a valuation tree.

$$(((p \land q) \rightarrow (\neg r)) \land (p \rightarrow q)) \rightarrow (p \rightarrow (\neg r)))$$
Exercise: Draw a valuation tree

Determine if the following formula is a tautology, a contradiction, or satisfiable but not a tautology, using a valuation tree.

\[ ((p \lor q) \leftrightarrow ((p \land \neg q) \lor (\neg p \land q))) \]
Getting an answer quickly

I found a valuation for which the formula is true. Does the formula have each property below?

- Tautology  YES  NO  MAYBE
- Contradiction  YES  NO  MAYBE
- Satisfiable  YES  NO  MAYBE

I found a valuation for which the formula is false. Does the formula have each property below?

- Tautology  YES  NO  MAYBE
- Contradiction  YES  NO  MAYBE
- Satisfiable  YES  NO  MAYBE
CQ: Getting an answer quickly
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