Propositional Logic: Semantics

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Lecture 5

Outline

Learning goals

By the end of this lecture, you should be able to

- Define a truth valuation.
- Determine the truth value of a formula given a truth valuation.
- Give a truth valuation under which a formula is true or false.
- Evaluate the truth value of a formula using a truth table and/or a valuation tree.
- Determine if a formula is a tautology, a contradiction, or satisfiable but not a tautology.

Outline

The meaning of well-formed formulas

How do we interpret the propositional language L^p and make the formulas express propositions?

Formulas are composed of atoms (proposition symbols) and connectives.

To interpret a formula, we have to give meanings to the atoms and the connectives.

Truth tables of connectives

The unary connective $\neg$:

A	$(\neg A)$
1	0
0	1

The binary connectives $\wedge$, $\vee$, $\rightarrow$, and $\leftrightarrow$:

A	B	$(A \wedge B)$	$(A \vee B)$	$(A \rightarrow B)$	$(A \leftrightarrow B)$
1	1	1	1	1	1
1	0	0	1	0	0
0	1	0	1	1	0
0	0	0	0	1	1

Definition of a truth valuation

Definition (Truth valuation)

A truth valuation is a function with the set of all proposition symbols as domain and $\{1, 0\}$ as range.

We use t to denote any truth valuation.

The value which t assigns to any formula A is written as A^t.

Values of formulas

The value assigned to formulas by a truth valuation t is defined by recursion:

$$p^t \in \{1, 0\}. \qquad (\neg A)^t = \begin{cases} 1, & \text{if } A^t = 0, \\ 0, & \text{otherwise.} \end{cases}$$

$$(A \wedge B)^t = \begin{cases} 1, & \text{if } A^t = B^t = 1, \\ 0, & \text{otherwise.} \end{cases} \qquad (A \vee B)^t = \begin{cases} 1, & \text{if } A^t = 1 \text{ or } B^t = 1, \\ 0, & \text{otherwise.} \end{cases}$$

$$(A \to B)^t = \begin{cases} 1, & \text{if } A^t = 0 \text{ or } B^t = 1, \\ 0, & \text{otherwise.} \end{cases} \qquad (A \leftrightarrow B)^t = \begin{cases} 1, & \text{if } A^t = B^t, \\ 0, & \text{otherwise.} \end{cases}$$

What is a truth valuation, intuitively?

Have you watched the TV series “Fringe”?

A truth valuation defines a parallel universe.

Our universe: The sky is blue and pigs do not fly.
Parallel universe 1: The sky is red, and pigs do not fly.
Parallel universe 2: The sky is blue, and pigs fly.

A structural induction example

Theorem: For any $A \in Form(L^p)$ and any truth valuation t, $A^t \in \{1, 0\}$.

Proof.

By induction on the structure of A. □

CQ Evaluating the truth value of a formula

Outline

Or, Exclusive Or, and Equivalence

A	B	$(A \vee B)$	$((A \wedge (\neg B)) \vee ((\neg A) \wedge B))$	$(A \leftrightarrow B)$
0	0	0	0	1
0	1	1	1	0
1	0	1	1	0
1	1	1	0	1

- Difference between the "or" and the exclusive or?
- Relationship between the exclusive or and the equivalence?

CQ Understanding an implication

Proving/disproving an implication

Consider the following implications.

(A) "If Alexandra is rich, she pays your tuition."

(B) $((p \wedge q) \rightarrow (p \vee q))$.

1. How do you prove that the implication is false?

2. How do you prove that the implication is true?

Review questions on the implication

- Think of an implication as a promise that someone made to you. In what case can you prove that the promise has been broken (i.e. the implication is false)?
- When the premise is true, what is the relationship between the truth value of the conclusion and the truth value of the implication?
- When the premise is false, the implication is vacuously true. Could you come up with an intuitive explanation for this?
- If the conclusion is true, is the implication true or false?
- The implication $(a \to b)$ is logically equivalent to $((\neg a) \lor b)$. Does this equivalent formula make sense to you? Explain.

Outline

Tautology, Contradiction, Satisfiable

A formula A is a tautology if and only if

For any truth valuation t, $A^t = 1$.

A formula A is a contradiction if and only if

For any truth valuation t, $A^t = 0$.

A formula A is satisfiable if and only if

There exists a truth valuation t such that $A^t = 1$.

Is a formula a tautology, contradiction, and/or satisfiable?

How do we determine if a formula is a tautology, a contradiction, and/or satisfiable?

Three approaches:

- Draw a truth table.
- Draw a valuation tree (a more compact truth table).
- Use reasoning to get an answer quickly.

Properties and truth tables

- Tautology: Each formula is
  true in EVERY/AT LEAST ONE/NO row of its truth table
  false in EVERY/AT LEAST ONE/NO row of its truth table.
- Satisfiable but not a tautology: Each formula is
  true in EVERY/AT LEAST ONE/NO row of its truth table
  false in EVERY/AT LEAST ONE/NO row of its truth table.
- Contradiction: Each formula is
  true in EVERY/AT LEAST ONE/NO row of its truth table
  false in EVERY/AT LEAST ONE/NO row of its truth table.

Simplifying a formula

After plugging in the value of a variable, we can simplify a formula as follows:

$$
\begin{array}{llll}
 & & p \to 1 \equiv & \\
p \land 1 \equiv & p \lor 1 \equiv & p \to 0 \equiv & p \leftrightarrow 1 \equiv \\
p \land 0 \equiv & p \lor 0 \equiv & 1 \to p \equiv & p \leftrightarrow 0 \equiv \\
p \land p \equiv & p \lor p \equiv & 0 \to p \equiv & p \leftrightarrow p \equiv \\
 & & p \to p \equiv &
\end{array}
$$

We can evaluate a formula by constructing a valuation tree using these rules.

A valuation tree

Prove that the following formula is a tautology using a valuation tree.

$$(((p \wedge q) \rightarrow (\neg r)) \wedge (p \rightarrow q)) \rightarrow (p \rightarrow (\neg r)))$$

Exercise: Draw a valuation tree

Determine if the following formula is a tautology, a contradiction, or satisfiable but not a tautology, using a valuation tree.

$$((p \vee q) \leftrightarrow ((p \wedge (\neg q)) \vee ((\neg p) \wedge q)))$$

Getting an answer quickly

I found a valuation for which the formula is true.
Does the formula have each property below?

- Tautology YES NO MAYBE
- Contradiction YES NO MAYBE
- Satisfiable YES NO MAYBE

I found a valuation for which the formula is false.
Does the formula have each property below?

- Tautology YES NO MAYBE
- Contradiction YES NO MAYBE
- Satisfiable YES NO MAYBE

CQ: Getting an answer quickly

Revisiting the Learning goals

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