Propositional Logic: Semantics

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Lecture 4

Outline

Admin stuff

FCC spectrum auction

- 20 billion revenue.
- Goal is to re-purpose radio spectrums.
- 2 auctions
- A computational problem in the buy back auction
- Satisfiability problems

Talk by Kevin Leyton-Brown
https://www.youtube.com/watch?v=u1-jJOivP70

Learning goals

By the end of this lecture, you should be able to

(Truth valuation, truth table, and valuation tree)

- Define a truth valuation.
- Determine the truth value of a formula given a truth valuation.
- Give a truth valuation under which a formula is true or false.
- Draw a truth table given a formula.
- Draw the valuation tree given a formula.

Learning goals (continued)

By the end of this lecture, you should be able to

(Properties of formulas)

- Define tautology, contradiction, and satisfiable formula.
- Determine if a given formula is a tautology, a contradiction, and/or a satisfiable formula.

The meaning of well-formed formulas

How do we interpret the propositional language L^p and make the formulas express propositions?

Formulas are composed of atoms (propositional symbols) and connectives.

To interpret a formula, we have to give meanings to the propositional symbols and the connectives.

Truth tables of connectives

The unary connective $\neg$:

A	$(\neg A)$
1	0
0	1

The binary connectives $\wedge$, $\vee$, $\to$, and $\leftrightarrow$:

A	B	$(A \wedge B)$	$(A \vee B)$	$(A \to B)$	$(A \leftrightarrow B)$
1	1	1	1	1	1
1	0	0	1	0	0
0	1	0	1	1	0
0	0	0	0	1	1

Definition of a truth valuation

A truth valuation is a function with the set of all propositional symbols as domain and $\{1, 0\}$ as range.

We use t to denote any truth valuation.

The value which t assigns to any formula A is written as A^t.

Values of formulas

The value assigned to formulas by a truth valuation t is defined by recursion:

$$p^t \in \{1, 0\}. \qquad (\neg A)^t = \begin{cases} 1, & \text{if } A^t = 0, \\ 0, & \text{otherwise.} \end{cases}$$

$$(A \wedge B)^t = \begin{cases} 1, & \text{if } A^t = B^t = 1, \\ 0, & \text{otherwise.} \end{cases} \qquad (A \vee B)^t = \begin{cases} 1, & \text{if } A^t = 1 \text{ or } B^t = 1, \\ 0, & \text{otherwise.} \end{cases}$$

$$(A \to B)^t = \begin{cases} 1, & \text{if } A^t = 0 \text{ or } B^t = 1, \\ 0, & \text{otherwise.} \end{cases} \qquad (A \leftrightarrow B)^t = \begin{cases} 1, & \text{if } A^t = B^t, \\ 0, & \text{otherwise.} \end{cases}$$

What is a truth valuation, intuitively?

Have you watched the TV series "Fringe"?

A truth valuation defines a parallel universe.

Our universe: The sky is blue and pigs do not fly.
Parallel universe 1: The sky is red, and pigs do not fly.
Parallel universe 2: The sky is blue, and pigs fly.

A structural induction example

Theorem: For any $A \in Form(L^p)$ and any truth valuation t, $A^t \in \{1, 0\}$.

Proof.

By induction on the structure of A. $\square$

CQs 14-15 Evaluating the truth value of a formula

Or, Exclusive Or, and Equivalence

A	B	$(A \vee B)$	$((A \wedge (\neg B)) \vee ((\neg A) \wedge B))$	$(A \leftrightarrow B)$
0	0	0	0	1
0	1	1	1	0
1	0	1	1	0
1	1	1	0	1

- Difference between the "or" and the exclusive or?
- Relationship between the exclusive or and the equivalence?

CQ 16 Understanding an implication

Assume that the following proposition is true.

If Alice is rich, she pays your tuition.

Assuming that Alice is rich, does she pay your tuition?

(A) Yes

(B) No

(C) Maybe

CQ 17 Understanding an implication

Assume that the following proposition is true.

If Alice is rich, she pays your tuition.

Assuming that Alice is not rich, does she pay your tuition?

(A) Yes

(B) No

(C) Maybe

Proving/disproving an implication

Consider the following implications.

(A) "If Alice is rich, she pays your tuition."

(B) $((p \land q) \to (p \lor q))$.

1. How do you prove that the implication is false?

2. How do you prove that the implication is true?

Review questions on the implication

- Think of an implication as a promise that someone made to you. In what case can you prove that the promise has been broken (i.e. the implication is false)?
- When the premise is true, what is the relationship between the truth value of the conclusion and the truth value of the implication?
- When the premise is false, the implication is vacuously true. Could you come up with an intuitive explanation for this?
- If the conclusion is true, is the implication true or false?
- The implication $(a \rightarrow b)$ is logically equivalent to $((\neg a) \vee b)$. Does this equivalent formula make sense to you? Explain.

Tautology, Contradiction, Satisfiable

A formula α is a tautology:

For every truth valuation t, $\alpha^t = \mathsf{T}$.

A formula α is a contradiction:

For every truth valuation t, $\alpha^t = \mathsf{F}$.

A formula α is satisfiable:

There exists a truth valuation t such that $\alpha^t = \mathsf{T}$.

Properties and truth tables

- Tautology: Each formula is
  true in EVERY/AT LEAST ONE/NO row of its truth table
  false in EVERY/AT LEAST ONE/NO row of its truth table.
- Satisfiable but not a tautology: Each formula is
  true in EVERY/AT LEAST ONE/NO row of its truth table
  false in EVERY/AT LEAST ONE/NO row of its truth table.
- Contradiction: Each formula is
  true in EVERY/AT LEAST ONE/NO row of its truth table
  false in EVERY/AT LEAST ONE/NO row of its truth table.

Is a formula a tautology, contradiction, and/or satisfiable?

Three approaches:

- Reasoning to get a quick answer
- Truth table
- Valuation tree: a more compact truth table

Reasoning to get a quick answer

I found a valuation for which the formula is true. Does the formula have each property below?

- Tautology YES NO MAYBE
- Contradiction YES NO MAYBE
- Satisfiable YES NO MAYBE

I found a valuation for which the formula is false. Does the formula have each property below?

- Tautology YES NO MAYBE
- Contradiction YES NO MAYBE
- Satisfiable YES NO MAYBE

CQ 21 Getting a quick answer

Simplifying a formula

Rather than filling out an entire truth table, we can simplify a formula in many situations:

$$
\begin{array}{lll}
 & & p \to \mathsf{T} \equiv \\
p \wedge \mathsf{T} \equiv & p \vee \mathsf{T} \equiv & p \to \mathsf{F} \equiv \\
p \wedge \mathsf{F} \equiv & p \vee \mathsf{F} \equiv & \mathsf{T} \to p \equiv \\
p \wedge p \equiv & p \vee p \equiv & \mathsf{F} \to p \equiv \\
 & & p \to p \equiv
\end{array}
$$

We can evaluate a formula by constructing a valuation tree using these rules.

A valuation tree

Show that $(((p \wedge q) \to (\neg r)) \wedge (p \to q)) \to (p \to (\neg r)))$ is a tautology by using a valuation tree.

Case 1: Suppose $t(p) = \mathrm{T}$.
The formula becomes $(((q \to (\neg r)) \wedge q) \to (\neg r))$.
If $t(q) = \mathrm{T}$, the formula is T (Check!).
If $t(q) = \mathrm{F}$, the formula is T (Check!).

Case 2: Suppose $t(p) = \mathrm{F}$. The formula is T. (Check!).

The formula is true for every valuation and is a tautology.

Note: We never had to consider the truth value of r in our analysis.

Additional exercises

Determine if each formula is a tautology, a contradiction, or satisfiable but not a tautology. Justify your answer.

1. $((((p \wedge q) \rightarrow r) \wedge (p \rightarrow q)) \rightarrow (p \rightarrow r))$
2. $(p \wedge (\neg p))$

Revisiting the Learning goals

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