Propositional Logic: Structural Induction

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Lecture 3

Outline

Learning goals

By the end of the lecture, you should be able to

- Prove properties of well-formed propositional formulas using structural induction.
- Prove properties of a recursively defined concept using structural induction.

Propositional language L^p

The propositional language L^p consists of three classes of symbols:

- Propositional symbols: p, q, r, $\dots$.
- Connective symbols: $\neg$, $\wedge$, $\vee$, $\rightarrow$, $\leftrightarrow$.
- Punctuation symbols: (and).

Well-formed propositional formulas

Definition ($Form(L^p)$)

An expression of L^p is a member of $Form(L^p)$ iff its being so follows from (1) - (3):

1. $Atom(L^p) \subseteq Form(L^p)$.
2. If $A \in Form(L^p)$, then $(\neg A) \in Form(L^p)$.
3. If $A, B \in Form(L^p)$, then $(A * B) \in Form(L^p)$.

Outline

Unique readability of well-formed formulas

Does every well-formed formula have a unique meaning? Yes.

Theorem: There is a unique way to construct each well-formed formula.

Properties of well-formed formulas

We may want to prove other properties of well-formed formulas.

- Every well-formed formula has at least one propositional variable.
- Every well-formed formula has an equal number of opening and closing brackets.
- Every proper prefix of a well-formed formula has more opening brackets than closing brackets.
- There is a unique way to construct every well-formed formula.

Why should you care?

Learning goals on structural induction:

- Prove properties of well-formed propositional formulas using structural induction.
- Prove properties of a recursively defined concept using structural induction.

Learning goals for future courses:

- Prove the space and time efficiency of recursive algorithms using induction.

Properties of well-formed formulas

Theorem: For every well-formed propositional formula φ, $P(\varphi)$ is true.

Recursive structure in well-formed formulas

Definition ($Form(L^p)$)

An expression of L^p is a member of $Form(L^p)$ iff its being so follows from (1) - (3):

1. $Atom(L^p) \subseteq Form(L^p)$. (Base case)
2. If $A \in Form(L^p)$, then $(\neg A) \in Form(L^p)$. (Inductive case)
3. If $A, B \in Form(L^p)$, then $(A * B) \in Form(L^p)$. (Inductive case)

A structural induction template for well-formed formulas

Theorem: For every well-formed formula φ, $P(\varphi)$ holds.
Proof by structural induction:

Base case: φ is a propositional symbol q. Prove that $P(q)$ holds.

Induction step:

Case 1: φ is $(\neg a)$, where a is well-formed.
Induction hypothesis: Assume that $P(a)$ holds.
We need to prove that $P((\neg a))$ holds.

Case 2: φ is $(a * b)$ where a and b are well-formed and $*$ is a binary connective.
Induction hypothesis: Assume that $P(a)$ and $P(b)$ hold.
We need to prove that $P((a * b))$ holds.

By the principle of structural induction, $P(\varphi)$ holds for every well-formed formula φ. QED

Review questions about the structural induction template

1. Why is the definition of a well-formed formula recursive?
2. To prove a property of well-formed formulas using structural induction, how many base cases and inductive cases are there in the proof?
3. In the base case, how do we prove the theorem? Does the proof rely on any additional assumption about the formula?
4. In an inductive case, how do we prove the theorem? Does the proof rely on any additional assumption about the formula?

Structural induction problems

Problem 1: Every well-formed formula has at least one propositional variable.

Problem 2: Every well-formed formula has an equal number of opening and closing brackets.

Problem 3: Every proper prefix of a well-formed formula has more opening brackets than closing brackets.

Outline

Structural induction for other problems

Structural induction is an important concept and it does not only apply to well-formed propositional formulas.

Let's look at some structural induction examples.

Ways of defining a set

- List all the elements in the set. Example: $A = \{1, 2, 3, 4\}$.
- Characterize the set by some property of all the elements in the set. Example: the set of even integers.
- Define a set inductively.

Inductively defined sets

An inductively definition of a set consists of three components:

- A domain set X
- A core set C
- A set of operations P

Closed and minimal set

A set Y is closed under a set of operations P iff applying any operation in P to elements in Y will always give us back an element in Y.

A set Y is a minimal set with respect to a property R if

- Y has property R, and
- For every set Z that has property R, $Y \subseteq Z$.

Defining a set inductively

Given a domain set X, a core set C and a set of operations P, $I(X, C, P)$ is the minimal subset of X that

- contains C, and
- is closed under P.

Example 1: Inductively defined sets

Consider the domain set, the core set, and the set of operations defined below.

- The domain set X $= \mathbb{R}$ (the set of real numbers)
- The core set C $= \{0\}$.
- The set of operations P $= \{f(x) = x + 1\}$

CQ: What set does this define?

(A) The set of natural numbers $\{0, 1, 2, ...\}$.

(B) The set of even natural numbers $\{0, 2, ...\}$.

(C) The set of integers $\{..., -2, -1, 0, 1, 2, ...\}$.

(D) The set of even integers $\{..., -2, 0, 2, ...\}$.

(E) The set of real numbers.

Example 2: Inductively defined sets

Consider the domain set, the core set, and the set of operations defined below.

- The domain set X = $\mathbb{R}$ (the set of real numbers)
- The core set C = $\{0, 2\}$.
- The set of operations P = $\{f1(x, y) = x + y, f2(x, y) = x - y\}$

CQ: What set does this define?

(A) The set of natural numbers $\{0, 1, 2, ...\}$.

(B) The set of even natural numbers $\{0, 2, ...\}$.

(C) The set of integers $\{..., -2, -1, 0, 1, 2, ...\}$.

(D) The set of even integers $\{..., -2, 0, 2, ...\}$.

(E) The set of real numbers.

Well-formed propositional formulas

Define the set of well-formed propositional formulas inductively.

- $X =$ the set of finite sequences of symbols in L^p.
- $C =$ the set of propositional variables.
- $P = \{f_1(x) = (\neg x), f_2(x, y) = (x * y)\}$
  where $*$ is one of $\wedge, \vee, \rightarrow, \leftrightarrow$.

Structural induction on $I(X, C, P)$

Claim: Every element of the set $I(X, C, P)$ has the property R.
Proof:

- Base case: Prove that R holds for every element in the core set C.
- Inductive case: Prove that for every operation $f \in P$ of arity k and any $y_1, ..., y_k \in I(X, C, P)$ such that $R(y_1), ..., R(y_k)$, $R(f(y_1, ..., y_k))$ holds.

Revisiting the learning goals

By the end of the lecture, you should be able to

- Prove properties of well-formed propositional formulas using structural induction.
- Prove properties of a recursively defined concept using structural induction.