Propositional Logic
Introduction and Syntax

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Lecture 2

Outline

Learning goals

Propositions and Connectives

Propositional Language

Revisiting the learning goals

Learning goals

By the end of the lecture, you should be able to

- Determine whether an English sentence is a proposition.
- Determine whether an English sentence is a simple or compound proposition.
- Determine whether a propositional formula is atomic and/or well-formed.
- Draw the parse tree of a well-formed propositional formula.
- Given a propositional formula with no parentheses, make it a well-formed formula by adding parentheses according to the precedence rules.

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Propositions

A proposition is a statement that is either true or false.

Meaningless statements, commands, and questions are not propositions.

CQ on proposition

Examples of propositions

- The sum of 3 and 5 is 8.
- The sum of 3 and 5 is 35.
- Goldbach's conjecture: Every even number greater than 2 is the sum of two prime numbers.

Examples of non-propositions

- Question: Where shall we go to eat?
- Command: Please pass the salt.
- Sentence fragment: The dogs in the park
- Non-sensical: Green ideas sleep furiously.
- Paradox: This sentence is false.

Compound and simple propositions

- A compound proposition is formed by means of logical connectives.

  The commonly used logical connectives are “not”, “and”, “or”, “if, then”, and “iff”.

- A simple proposition is not compound and cannot be further divided.

Interpreting a compound proposition

To interpret a compound proposition, we need to understand the meanings of the connectives.

Let A and B be arbitrary propositions.

We will use 1 and 0 to denote true and false respectively.

Negation

"Not A" is true if and only if A is false.

A	not A
1	0
0	1

Conjunction

"A and B" is true if and only if both A and B are true.

A	B	A and B
1	1	1
1	0	0
0	1	0
0	0	0

Disjunction

A	B	A or B
1	1	1
1	0	1
0	1	1
0	0	0

"Or" may be interpreted in two ways

- The inclusive sense of "A or B or both"
- The exclusive sense of "A or B but not both"

In mathematics, the inclusive sense of "or" is commonly used.

Implication

A	B	if A then B
1	1	1
1	0	0
0	1	1
0	0	1

The only circumstance in which "if A then B" is false is when A is true and B is false.

The truth of "if A then B" enables us to

- infer the truth of B from the truth of A, and
- infer nothing from the falsehood of A.

Whenever A is false, "if A then B" is vacuously true.

The verification of "if A then B" does not require doing anything to deduce B from A.

Equivalence

"A iff B" is the same as "if A then B, and if B then A".

iff is pronounced as if and only if.

A	B	A iff B
1	1	1
1	0	0
0	1	0
0	0	1

CQ on compound or simple propositions

Negated statements are compound propositions, not atomic propositions.

Choose your atomic propositions to be the positive statements (i.e. without any embedded negations).

Remarks on connectives

The arity of a connective:

- The negation is a unary connective. It only applies to one proposition.
- All other connectives are binary connectives. They apply to two propositions.

Is a connective symmetric?

- And, Or, and Equivalence are symmetric. The order of the two propositions does not affect the truth value of the compound proposition.
- Implication is not symmetric. If A then B, and if B then A have different truth values.

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Propositional language L^p

The propositional language L^p consists of three classes of symbols:

- Proposition symbols: p, q, r, ...
- Connective symbols: $\neg$, $\wedge$, $\vee$, $\rightarrow$, $\leftrightarrow$.

Oral reading of logical connectives

$\neg$	not	negation
$\wedge$	and	conjunction
$\vee$	or	(inclusive) disjunction
$\rightarrow$	if, then (imply)	implication
$\leftrightarrow$	iff (equivalent to)	equivalence

- Punctuation symbols: (and).

Expressions of L^p

- expressions are finite strings of symbols. Examples: p, pq, (r), $p\wedge \rightarrow q$ and $\neg(p \wedge q)$.
- The length of an expression is the number of occurrences of symbols in it.
- empty expression: an expression of length 0, denoted by λ.
- two expressions u and v are equal if they are of the same length and have the same symbols in the same order.
- an expression is read from left to right.

Expression terminologies

- uv denotes the result of concatenating two expressions u, v in this order. Note that $\lambda u = u\lambda = u$.
- v is a segment of u if $u = w_1 v w_2$ where u, v, w_1, w_2 are expressions.

  v is a proper segment of u if v is non-empty and $v \neq u$.

  If $u = vw$, where u, v, w are expressions, then v is an initial segment (prefix) of u. Similarly, w is a terminal segment (suffix) of u.

Atomic formulas

Definition ($Atom(L^p)$)

$Atom(L^p)$ is the set of expressions of L^p consisting of a proposition symbol only.

Well-formed propositional formulas

Definition ($Form(L^p)$)

An expression of L^p is a member of $Form(L^p)$ if and only if its being so follows from (1) - (3):

1. $Atom(L^p) \subseteq Form(L^p)$.
2. If $A \in Form(L^p)$, then $(\neg A) \in Form(L^p)$.
3. If $A, B \in Form(L^p)$, then $(A * B) \in Form(L^p)$ where $*$ is one of the four binary connectives.

Note that $Form(L^p)$ is the minimum set that satisfies the three conditions above.

CQ on the first symbol in a well-formed formula

CQ on well-formed propositional formulas

Example: Generating Formulas

The following expression is a formula.

$$((p \lor q) \to ((\neg p) \leftrightarrow (q \land r)))$$

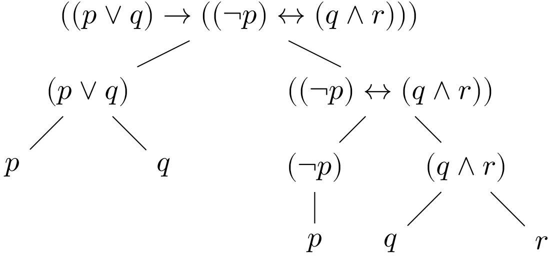
How is it generated using the definition of well-formed propositional formulas? One can use parse trees to analyze formulas.

Example: Parse Tree

Draw the parse tree for the following formula.

$$((p \vee q) \rightarrow ((\neg p) \leftrightarrow (q \wedge r)))$$

Parse tree:



Exercise: Parse Trees

Draw the parse tree for the following formula.

$$(((\neg p) \land q) \to (p \land (q \lor (\neg r)))$$

Precedence rules: for humans

Consider the following sequence of connectives:

$$\neg, \land, \lor, \to, \leftrightarrow$$

Each connective on the left has priority over those on the right.

Examples: Add back the brackets based on the precedence rules.

- $\neg p \lor q$
- $p \land q \lor r$
- $p \to q \leftrightarrow p$
- $\neg p \to p \land \neg q \lor r \leftrightarrow q$

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