Propositional Logic
Introduction and Syntax

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Lecture 2
Outline

- Learning goals
- Propositions and Connectives
- Propositional Language
- Revisiting the learning goals
Learning goals

By the end of the lecture, you should be able to

▶ Determine whether an English sentence is a proposition.
▶ Determine whether an English sentence is a simple or compound proposition.
▶ Determine whether a propositional formula is atomic and/or well-formed.
▶ Draw the parse tree of a well-formed propositional formula.
▶ Given a propositional formula with no parentheses, make it a well-formed formula by adding parentheses according to the precedence rules.
Outline

Learning goals

Propositions and Connectives

Propositional Language

Revisiting the learning goals
A proposition is a statement that is either true or false.

Meaningless statements, commands, and questions are not propositions.
CQ on proposition
Examples of propositions

- The sum of 3 and 5 is 8.
- The sum of 3 and 5 is 35.
- Goldbach’s conjecture: Every even number greater than 2 is the sum of two prime numbers.
Examples of non-propositions

- Question: Where shall we go to eat?
- Command: Please pass the salt.
- Sentence fragment: The dogs in the park
- Non-sensical: Green ideas sleep furiously.
- Paradox: This sentence is false.
A compound proposition is formed by means of logical connectives.

The commonly used logical connectives are “not”, “and”, “or”, “if, then”, and “iff”.

A simple proposition is not compound and cannot be further divided.
Interpreting a compound proposition

To interpret a compound proposition, we need to understand the meanings of the connectives.

Let $A$ and $B$ be arbitrary propositions.

We will use 1 and 0 to denote true and false respectively.
Negation

“Not A” is true if and only if A is false.

<table>
<thead>
<tr>
<th>$A$</th>
<th>not $A$</th>
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<tbody>
<tr>
<td>1</td>
<td>0</td>
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<tr>
<td>0</td>
<td>1</td>
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Conjunction

“A and B” is true if and only if both A and B are true.

<table>
<thead>
<tr>
<th>A</th>
<th>B</th>
<th>A and B</th>
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<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>1</td>
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<tr>
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Disjunction

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<tr>
<td>$A$</td>
<td>$B$</td>
<td>$A$ or $B$</td>
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<td>1</td>
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<td>1</td>
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<td>1</td>
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“Or” may be interpreted in two ways

- The inclusive sense of “A or B or both”
- The exclusive sense of “A or B but not both”

In mathematics, the inclusive sense of “or” is commonly used.
Implication

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<tr>
<th>$A$</th>
<th>$B$</th>
<th>if $A$ then $B$</th>
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<tbody>
<tr>
<td>1</td>
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The only circumstance in which “if $A$ then $B$” is false is when $A$ is true and $B$ is false.

Whenever $A$ is false, “if $A$ then $B$” is vacuously true.
"A iff B" is the same as "if A then B, and if B then A". iff is pronounced as if and only if.

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<th>A iff B</th>
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CQ on compound or simple propositions
Remarks on connectives

The arity of a connective:

▶ The negation is a unary connective. It only applies to one proposition.
▶ All other connectives are binary connectives. They apply to two propositions.

Is a connective symmetric?

▶ And, Or, and Equivalence are symmetric. The order of the two propositions does not affect the truth value of the compound proposition.
▶ Implication is not symmetric. If A then B, and if B then A have different truth values.
Outline

Learning goals

Propositions and Connectives

Propositional Language

Revisiting the learning goals
The propositional language $L^p$ consists of three classes of symbols:

- **Proposition symbols:** $p, q, r, \ldots$
- **Connective symbols:** $\neg, \land, \lor, \rightarrow, \leftrightarrow$.

**Oral reading of logical connectives**


- $\neg$ not
- $\land$ and
- $\lor$ or
- $\rightarrow$ if, then (imply)
- $\leftrightarrow$ iff (equivalent to)

- **Punctuation symbols:** ( and ).
Expressions of $L^p$

- **expressions** are finite strings of symbols. Examples: $p$, $pq$, $(r)$, $p \land \rightarrow q$ and $\neg(p \land q)$.
- The **length** of an expression is the number of occurrences of symbols in it.
- **empty expression**: an expression of length 0, denoted by $\lambda$.
- two expressions $u$ and $v$ are **equal** if they are of the same length and have the same symbols in the same order.
- an expression is read from left to right.
Expression terminologies

- $uv$ denotes the result of concatenating two expressions $u$, $v$ in this order. Note that $\lambda u = u \lambda = u$.

- $v$ is a segment of $u$ if $u = w_1v w_2$ where $u, v, w_1, w_2$ are expressions.

- $v$ is a proper segment of $u$ if $v$ is non-empty and $v \neq u$.

If $u = vw$, where $u, v, w$ are expressions, then $v$ is an initial segment (prefix) of $u$. Similarly, $w$ is a terminal segment (suffix) of $u$. 
Definition \((\text{Atom}(L^p))\)

\(\text{Atom}(L^p)\) is the set of expressions of \(L^p\) consisting of a proposition symbol only.
Well-formed propositional formulas

Definition ($\text{Form}(L^p)$)

An expression of $L^p$ is a member of $\text{Form}(L^p)$ if and only if its being so follows from (1) - (3):

1. $\text{Atom}(L^p) \subseteq \text{Form}(L^p)$.
2. If $A \in \text{Form}(L^p)$, then $(\neg A) \in \text{Form}(L^p)$.
3. If $A, B \in \text{Form}(L^p)$, then $(A \ast B) \in \text{Form}(L^p)$ where $\ast$ is one of the four binary connectives.

Note that $\text{Form}(L^p)$ is the minimum set that satisfies the three conditions above.
CQ on the first symbol in a well-formed formula
CQ on well-formed propositional formulas
The following expression is a formula.

\[((p \lor q) \rightarrow ((\neg p) \leftrightarrow (q \land r)))\]

How is it generated using the definition of well-formed propositional formulas? One can use parse trees to analyze formulas.
Example: Parse Tree

Draw the parse tree for the following formula.

\[((p \lor q) \rightarrow ((\neg p) \leftrightarrow (q \land r)))\]

Parse tree:
Exercise: Parse Trees

Draw the parse tree for the following formula.

$$(((\neg p) \land q) \rightarrow (p \land (q \lor (\neg r))))$$
Precedence rules: for humans

Consider the following sequence of connectives:

\[ \neg, \wedge, \vee, \rightarrow, \leftrightarrow \]

Each connective on the left has priority over those on the right.

Examples: Add back the brackets based on the precedence rules.

- \( \neg p \vee q \)
- \( p \wedge q \vee r \)
- \( p \rightarrow q \leftrightarrow p \)
- \( \neg p \rightarrow p \wedge \neg q \vee r \leftrightarrow q \)
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