Reversing an array.                                                    Thu Nov. 23

Consider an array $R$ with $n$ integers. $R[1], R[2], \ldots, R[n]$

We want to reverse the order of its elements.

Algorithm:

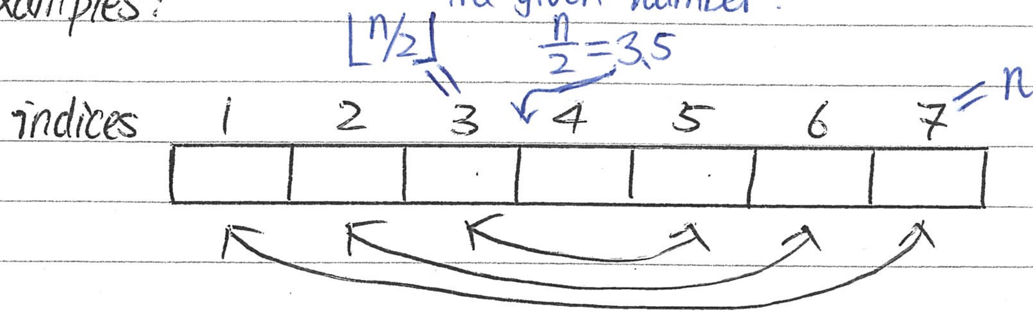
for each $1 \le j \le \lfloor n/2 \rfloor$

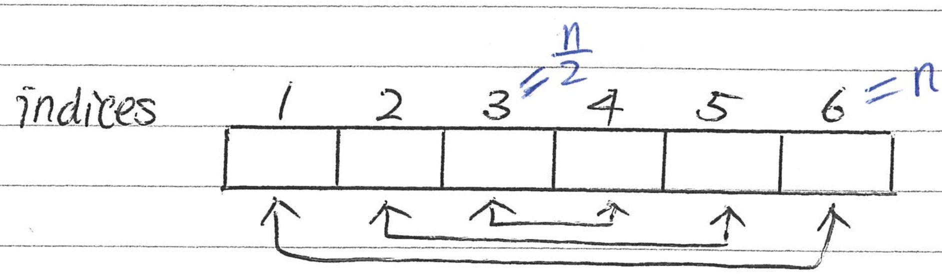
swap $R[j]$ and $R[n+1-j]$.

round down to the nearest integer.

floor: returns the largest integer that is smaller than the given number.

Examples:




$1 + 6 = 7 = n+1$

$2 + 5 = 7 = n+1$

$3 + 4 = 7 = n+1$

If we are swapping $R[j]$ and $R[k]$, then $j+k = n+1$,

or $k = n+1-j$.

Solutions

$r_x$: the $x^{th}$ element of the array R at the start of the program execution.

Reversing an Array.

$$\langle\!| ((n \geq 0) \wedge (\forall x ((1 \leq x \leq n) \rightarrow (R[x] = \underbrace{r_x}_{\text{logical variables}})))) |\!\rangle$$

- do not appear in the program.

```
j = 1;
→ while (j <= floor(n/2)) {
    t = R[j];
    R[j] = R[n+1-j];
    R[n+1-j] = t;
    j = j + 1;
}
```

$$\langle\!| (\forall x ((1 \leq x \leq n) \rightarrow (R[x] = r_{n+1-x}))) |\!\rangle$$

solutions

$\lfloor n/2 \rfloor = \lfloor 7/2 \rfloor = \lfloor 3.5 \rfloor = 3$

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| n | j | Indices | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|---|---|
| 7 | 1 | | $r_1$ | $r_2$ | $r_3$ | $\boxed{r_4}$ | $r_5$ | $r_6$ | $r_7$ |
| 7 | 2 | | $r_7$ | $r_2$ | $r_3$ | $\boxed{r_4}$ | $r_5$ | $r_6$ | $r_1$ |
| | | | $\underbrace{1 \le x \le j-1}$ | | $\underbrace{j \le x \le \lfloor n/2 \rfloor}$ | | $\underbrace{n+1-\lfloor n/2 \rfloor \le x \le n+1-j}$ | $\underbrace{n+1-j+1 \le x \le n}$ | |
| 7 | 3 | | $r_7$ | $r_6$ | $r_3$ | $\boxed{r_4}$ | $r_5$ | $r_2$ | $r_1$ |
| 7 | 4 | | $r_7$ | $r_6$ | $r_5$ | $\boxed{r_4}$ | $r_3$ | $r_2$ | $r_1$ |

$$
\begin{aligned}
Inv'(j) = \\
(\forall x\ & ((1 \le x \le j-1) \rightarrow (R[x] = r_{n+1-x})) \wedge \\
& ((j \le x \le \lfloor n/2 \rfloor) \rightarrow (R[x] = r_x)) \wedge \\
& ((n+1-\lfloor n/2 \rfloor \le x \le n+1-j) \rightarrow (R[x] = r_x)) \wedge \\
& ((n+1-j+1 \le x \le n) \rightarrow (R[x] = r_{n+1-x})))
\end{aligned}
$$

It only makes sense for $j \le (\lfloor n/2 \rfloor + 1)$

Our invariant:

$$Inv(j) = Inv'(j) \wedge (j \le (\lfloor n/2 \rfloor + 1))$$

solutions

Reversing an Array.

$\lhd ((n \geq 0) \wedge (\forall x ((1 \leq x \leq n) \rightarrow (R[x] = r_x)))) \rhd$

$\lhd Inv(1) \rhd$ implied (a)

j=1;

$\lhd Inv(j) \rhd$ assignment

while (j <= floor(n/2)) {

$\lhd (Inv(j) \wedge (j \leq \lfloor n/2 \rfloor)) \rhd$ partial-while

$\lhd Inv(j+1)[R\{j \leftarrow R[n+1-j]\}\{(n+1-j) \leftarrow R[j]\}/R] \rhd$ implied. (b)

t = R[j];

$\lhd Inv(j+1)[R\{j \leftarrow R[n+1-j]\}\{(n+1-j) \leftarrow t\}/R] \rhd$ assignment

R[j] = R[n+1-j];

$\lhd Inv(j+1)[R\{(n+1-j) \leftarrow t\}/R]$ array assignment

R[n+1-j] = t;

$\lhd Inv(j+1) \rhd$ array assignment

j = j+1;

$\lhd Inv(j) \rhd$ assignment

}

$\lhd (Inv(j) \wedge (j > \lfloor n/2 \rfloor)) \rhd$ partial-while

$\lhd (\forall x ((1 \leq x \leq n) \rightarrow (R[x] = r_{n+1-x}))) \rhd$ implied. (c)

solutions

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Implied (a):

The conclusion: $Inv(1) \equiv Inv'(1) \wedge (1 \le 1 \le (\lfloor n/2 \rfloor + 1))$

$$
\begin{aligned}
Inv'(1) \equiv\ & \cancel{((1 \le x \le 0) \rightarrow (R[x] = r_{n+1-x}))} \wedge \\
& ((1 \le x \le \lfloor n/2 \rfloor) \rightarrow (R[x] = r_x)) \wedge \\
& ((n+1-\lfloor n/2 \rfloor \le x \le n) \rightarrow (R[x] = r_x)) \wedge \\
& \cancel{((n+1 \le x \le n) \rightarrow (R[x] = r_{n+1-x}))}.
\end{aligned}
$$

$$
\begin{aligned}
\equiv\ & ((1 \le x \le \lfloor n/2 \rfloor) \rightarrow (R[x] = r_x)) \wedge \\
& ((n+1-\lfloor n/2 \rfloor \le x \le n) \rightarrow (R[x] = r_x))
\end{aligned}
$$

For every $x$, the $x^{th}$ element is equal to $r_x$.

The premise:

$(n \ge 0) \wedge (\forall x\ ((1 \le x \le n) \rightarrow (R[x] = r_x)))$

For every $x$, the $x^{th}$ element is equal to $r_x$.

The premise is the same as the conclusion, so the implication holds.

solutions

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Implied (b):

The premise: $Inv(j) \land (j \leq \lfloor n/2 \rfloor)$

The conclusion:

$$Inv(j+1)[\underbrace{R\{j \leftarrow R[n+1-j]\}\{(n+1-j) \leftarrow R[j]\}}_{R'}/R]$$

Unpacking the conclusion:

$Inv'(j+1) \equiv$

$$((1 \leq x \leq j) \rightarrow (R'[x] = r_{n+1-x})) \land$$
$$((j+1 \leq x \leq \lfloor n/2 \rfloor) \rightarrow (R'[x] = r_x)) \land$$
$$((n+1-\lfloor n/2 \rfloor \leq x \leq n-j) \rightarrow (R'[x] = r_x)) \land$$
$$((n-j+1 \leq x \leq n) \rightarrow (R'[x] = r_{n+1-x}))$$

$R'$ and $R$ differ in only the $j^{th}$ and $(n+1-j)^{th}$ elements.

Based on $Inv'(j+1)$:

$$R'[j] = r_{n+1-j} \quad \text{and} \quad R'[n+1-j] = r_{n+1-(n+1-j)} = r_j.$$

Unpacking the premise:

$Inv(j) \equiv$

$$1 \leq x \leq j-1 \rightarrow R[x] = r_{n+1-x}$$
$$j \leq x \leq \lfloor n/2 \rfloor \rightarrow R[x] = r_x$$
$$n+1-\lfloor n/2 \rfloor \leq x \leq n+1-j \rightarrow R[x] = r_x$$
$$n+1-j+1 \leq x \leq n \rightarrow R[x] = r_{n+1-x}.$$

Based on $Inv(j)$:

$$R[j] = r_j \qquad R[n+1-j] = r_{n+1-j}.$$

The premise: $(R[j] = r_j) \land (R[n+1-j] = r_{n+1-j})$

The conclusion: $(R'[j] = r_{n+1-j}) \land (R'[n+1-j] = r_j)$

We need to show that ① $R[j] = R'[n+1-j]$ ② $R[n+1-j] = R'[j]$.

Recall that $R' = R\{j \leftarrow R[n+1-j]\}\{(n+1-j) \leftarrow R[j]\}$

So we need to show that

① $R[j] = R\{j \leftarrow R[n+1-j]\}\{(n+1-j) \leftarrow R[j]\}[n+1-j]$

② $R[n+1-j] = R\{j \leftarrow R[n+1-j]\}\{(n+1-j) \leftarrow R[j]\}[j]$.

sols.

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Implied (c):

The premise : $Inv(j) \wedge (j > \lfloor n/2 \rfloor)$

The conclusion : $(\forall x\ ((1 \leq x \leq n) \rightarrow (R[x] = r_{n+1-x})))$

For every $x$, the $x^{th}$ element is equal to $r_{n+1-x}$.

The premise :

$$Inv'(j) \wedge (\cancel{1 \leq}\ j \leq (\lfloor n/2 \rfloor + 1)) \wedge (\underbrace{j > \lfloor n/2 \rfloor}_{j \geq \lfloor n/2 \rfloor + 1})$$

$$\equiv Inv'(j) \wedge (j = (\lfloor n/2 \rfloor + 1)).$$

$$\equiv$$

$$((1 \leq x \leq \lfloor n/2 \rfloor) \rightarrow (R[x] = r_{n+1-x})) \wedge$$
$$\cancel{((\lfloor n/2 \rfloor + 1 \leq x \leq \lfloor n/2 \rfloor) \rightarrow (R[x] = r_x)) \wedge}$$
$$\cancel{((n+1-\lfloor n/2 \rfloor \leq x \leq n - \lfloor n/2 \rfloor) \rightarrow (R[x] = r_x)) \wedge}$$
$$((n - \lfloor n/2 \rfloor + 1 \leq x \leq n) \rightarrow (R[x] = r_{n+1-x})).$$

$$\equiv$$

$$((1 \leq x \leq \lfloor n/2 \rfloor) \rightarrow (R[x] = r_{n+1-x})) \wedge$$
$$((n - \lfloor n/2 \rfloor + 1 \leq x \leq n) \rightarrow (R[x] = r_{n+1-x}))$$

For every $x$, the $x^{th}$ element is equal to $r_{n+1-x}$.

solutions