Semantic entailment. Sept 26.

Show that $\{(p \to q), (q \to r)\} \models (p \to r)$.

Proof ①:

p	q	r	$p \to q$	$q \to r$	$p \to r$	
0	0	0	1	1	1	*
0	0	1	1	1	1	*
0	1	0	1	0	1	
0	1	1	1	1	1	*
1	0	0	0	1	0	
1	0	1	0	1	1	
1	1	0	1	0	0	
1	1	1	1	1	1	*

The * marks all the rows in which $p \to q$ and $q \to r$ are both true. $(p \to r)$ is true in all of the * rows, So the entailment holds. QED.

Proof ②: We prove this by contradiction.
Assume that the entailment does not hold.
There is a truth valuation t such that
$(p \to q)^t = T$, $(q \to r)^t = T$ and $(p \to r)^t = F$.

If $(p \to r)^t = F$, then it has to be that $p^t = T$ and $r^t = F$.

$$\left.\begin{array}{l}\text{If } (p \to q)^t = T \text{ and } p^t = T, \text{ then } q^t = T. \\ \text{If } (q \to r)^t = T \text{ and } r^t = F, \text{ then } q^t = F\end{array}\right\} \text{This is a contradiction.}$$

Our assumption is false and the entailment holds.

QED.

Semantic entailment

Sept 26.

Show that

$$\{((p \to (\neg q)) \vee r), (q \wedge (\neg r)), (p \leftrightarrow r)\} \not\vDash (p \wedge (q \to r))$$

Proof: Consider a truth valuation t such that

$$p^t = F, \quad q^t = T, \text{ and } r^t = F.$$

$$((p \to (\neg q)) \vee r)^t = ((F \to F) \vee F) = T$$

$$(q \wedge (\neg r))^t = (T \wedge T) = T$$

$$(p \leftrightarrow r)^t = (F \leftrightarrow F) = T$$

$$(p \wedge (q \to r))^t = (F \wedge (T \to F)) = F$$

The premises are true but the conclusion is false, so the entailment does not hold.

QED.