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Theorem: $\{\wedge, \vee\}$ is not an adequate set of connectives.

Proof: To prove this theorem, we will prove the following lemma first.

Lemma: For any formula which uses only $\wedge$ and $\vee$ as connectives, if every variable in the formula is true, then the formula is true.

See the next page for a proof of this lemma using structural induction. Now we use this lemma below.

It's sufficient to show that we cannot write $(\neg x)$ using only $\wedge$ and $\vee$.

Assume x is true. Then $(\neg x)$ is false.
By the lemma, for any formula using x, $\wedge$, and $\vee$, the formula must be true when x is true. Therefore, we cannot use x, $\wedge$ and $\vee$ to write a formula that is false when x is true

So we cannot write $(\neg x)$ using only $\wedge$ and $\vee$.
$\{\wedge, \vee\}$ is not an adequate set of connectives.

QED.

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$\{\wedge, \vee\}$ is not an adequate set of connectives.

Lemma: For any formula which uses only $\wedge$ and $\vee$ as connectives, if every variable in the formula is true, then the formula is true.

Let $P(\varphi)$ denote φ is true when every variable in φ is true where φ uses only $\wedge$ and $\vee$ as connectives.

Proof by structural induction on φ.

Base case: φ is a propositional variable p.
If p is true, then φ is true. $P(\varphi)$ holds.

Induction step:

case 1: $\varphi = (x \wedge y)$ where x and y use only $\wedge$ and $\vee$ as connectives.

Induction Hypothesis: if every variable in x (and y) is true, then x (and y) is true.

Assume every variable in φ is true. Then every variable in x and y is true. By induction hypothesis, x and y are true. So $\varphi \equiv (T \wedge T) \equiv T$.

case 2: $\varphi = (x \vee y)$ where x and y use only $\wedge$ and $\vee$ as connectives.

Induction Hypothesis: if every variable in x (and y) is true, then x (and y) is true.

Assume every variable in φ is true. Then every variable in x and y is true. By induction hypothesis, x and y are true. So $\varphi \equiv (T \vee T) \equiv T$.

By the principle of structural induction, $P(\varphi)$ holds for every φ that uses only $\wedge$ and $\vee$ as connectives.

QED