Natural deduction rules

① conjunction rules

(premises). formulas that we assume to be true.

$$\frac{x \quad y}{x \wedge y}\ \wedge i \qquad \text{inference line} \to \frac{\overbrace{x \wedge y}}{\underbrace{x}} \qquad \frac{x \wedge y}{y}\ \wedge e$$

conclusion. what we can derive to be true

② implication rules

$$\text{subproof} \to \dfrac{\boxed{\begin{array}{c} x \to \text{assumption} \\ \vdots \\ y \to \text{conclusion.} \end{array}}}{x \to y}\ \to i \qquad \frac{\begin{array}{c} x \\ x \to y \end{array}}{y}\ \to e$$

(modus ponens)

③ disjunction rules

$$\frac{x}{x \vee y} \qquad \frac{y}{x \vee y}\ \vee i \qquad \dfrac{x \vee y \quad \boxed{\begin{array}{c} x \\ \vdots \\ z \end{array}}^{\,x \to z} \quad \boxed{\begin{array}{c} y \\ \vdots \\ z \end{array}}^{\,y \to z}}{z}\ \vee e \qquad \frac{\begin{array}{c} x \vee y \\ x \to z \\ y \to z \end{array}}{z}$$

(proof by cases)

④ negation rules ⊥ also called "bottom"

$$\dfrac{\boxed{\begin{array}{c} x \\ \vdots \\ \text{contradiction F (false)} \to \perp \to \perp \end{array}}}{(\neg x)}\ \neg i \qquad \frac{\begin{array}{c} x \\ (\neg x) \end{array}}{\perp}\ \neg e(\perp i)$$

(proof by contradiction)

two names

⑤ contradiction rules

$$\frac{\begin{array}{c} x \\ (\neg x) \end{array}}{\perp}\ \perp i(\neg e) \qquad \frac{\perp}{x}\ \perp e$$

$\perp \to$ False (we can infer ~~derive~~ anything from false/contradiction.

⑥ double negation rules (this is also a logical identity)

$$\frac{x}{\neg\neg x}\ \neg\neg i \qquad \frac{\neg\neg x}{x}\ \neg\neg e$$

Natural deduction rules.

A few notes about subproofs.

- It's a small proof inside a big proof.
- Inside a subproof,
  - We can start a subproof with any formula we want. (called an assumption). (The assumption doesn't need to have appeared earlier in the proof.)
  - We can use any formula that has appeared before, inside and/or outside the subproof.
- Outside a subproof:
  - No singe formula can escape a subproof. We cannot take a formula inside a subproof and use it later.
  - We have to use the subproof as a whole.

An example of or-elimination $\vee e$:

I eat an apple or I eat an orange.	$a \vee o$
If I eat an apple, then I am happy.	$a \to h$
If I eat an orange, then I am happy.	$o \to h$
Therefore, I am happy.	h

This inference is valid. You can't be sure whether I ate an apple or an orange, but either way, I am happy.