Assignment 4 Q1(c) $\phi \vdash (((P \to Q) \to P) \to P)$

① Always start a proof on a piece of scratch paper by writing down all the premises and the conclusion.

Proof: (no premise, so nothing to write down)

$\vdots$

(leave ample space)

$\vdots$

$(((P \to Q) \to P) \to P)$ ← (write down the conclusion).

② Look at the premises. Can we apply any (elimination) rule to simplify them?

(nothing to do because we have no premise.)

③ Look at the conclusion. How can we reach it using any rule? (most likely an introduction rule.)

(3a) $(((P \to Q) \to P) \to P)$ is an implication. So we could reach it by using $\to i$. Our proof becomes:

Proof:

1	$((P \to Q) \to P)$	assumption.
	$\vdots$	
	P	
	$(((P \to Q) \to P) \to P)$	$\to i$: 1–

Note that I set up $\underbrace{\text{the structure of the subproof}}$ first before filling in its contents.

the first line as the assumption
the last line in the box
the rule using the subproof. ($\to i$)

③b We could also reach the conclusion by $\neg i$ and $\neg\neg e$. Our proof becomes:

Proof:

$(\neg(((P \to q) \to P) \to P))$ assumption

$\vdots$

$\bot$

$(\neg(\neg(((P \to q) \to P) \to P)))$ $\neg i$

$(((P \to q) \to P) \to P)$ $\neg\neg e$

For now, let's go with ③a.

④a Inside the subproof of ③a,

- look at the assumption $((P \to q) \to P)$. Can we apply any rule to simplify it or use it?

  We could try eliminating the $\to$ but we don't know that $(P \to q)$ is true.
  Cannot think of ways of using the assumption

- Look at the conclusion P, How can we reach it with any rule?

  We could try $\neg i$ and $\neg\neg e$ again.

Proof:

1 $((P \to q) \to P)$ assumption

2 $(\neg P)$ assumption

$\vdots$

$\bot$

$(\neg(\neg P))$ $\neg i$

P $\neg\neg e$:

$(((P \to q) \to P) \to P)$ $\to i$: 1–

In the inner box, to reach a contradiction $\bot$, we need to derive either $(\neg((P \to q) \to P))$ or P.

Deriving $(\neg((P \to q) \to P))$ would require using $\neg i$ again and assuming $((P \to q) \to P)$ again in another inner box. We seem to be running in circles with this approach.

(4a) continued. Let's try deriving p to reach $\bot$.

Proof:

1	$((p \to q) \to p)$	assumption
2	$(\neg p)$	assumption
	$\vdots$	
	p	
	$\bot$	$\bot i$
	$(\neg(\neg p))$	$\neg i$
	p	$\neg\neg e$
	$(((p \to q) \to p) \to p)$	$\to i$

How do we conclude p in the inner subproof?
We know $((p \to q) \to p)$ and $(\neg p)$ in there.
If we can prove $(p \to q)$, then we can use $\to e$ on $(p \to q)$ and $((p \to q) \to p)$ to conclude p. Let's try this

Proof:

1	$((p \to q) \to p)$	assumption
2	$(\neg p)$	assumption
	$\vdots$	
	$(p \to q)$	
	p	$\to e$
	$\bot$	$\bot i$
	$(\neg(\neg p))$	$\neg i$
	p	$\neg\neg e$
	$(((p \to q) \to p) \to p)$	$\to i$

(5a) To reach $(p \to q)$, we could use the $\to i$ rule by assuming p and deriving q.

Also, check our intuition. If we assume $(\neg p)$ or p is false, does it make sense that $(p \to q)$ is true? Sure, if p is false, $p \to q$ is vacuously true.

The next version of the proof is on the next page.

5a continued.

Proof:

1	$((p \to q) \to p)$	assumption
2	$(\neg p)$	assumption
3	p	assumption
4.	$\vdots$	
	q	
	$(p \to q)$	$\to i$:
	p	$\to e$:
	$\perp$	
	$(\neg(\neg p))$	$\neg i$:
	p	$\neg\neg e$:
	$(((p \to q) \to p) \to p)$	$\to i$:

How do we fill in the subproof starting from line 3?
- On line 4, knowing $(\neg p)$ and p, we can derive $\perp$ by $\perp i$.
- Then we can derive q from $\perp$ by $\perp e$.

This completes our proof, as shown below.

Proof:

1	$((p \to q) \to p)$	assumption
2	$(\neg p)$	assumption
3	p	assumption
4	$\perp$	$\perp i$: 2,3
5	q	$\perp e$: 4
6	$(p \to q)$	$\to i$: 3–5
7	p	$\to e$: 1,6
8	$\perp$	$\perp i$: 2,7
9	$(\neg(\neg p))$	$\neg i$: 2–8
10	p	$\neg\neg e$: 9
11	$(((p \to q) \to p) \to p)$	$\to i$: 1–10

Finally, we complete the proof by filling out the justification and the line numbers.