# Soundness and Completeness of Natural Deduction.

Soundness: If $\Sigma \vdash \alpha$, then $\Sigma \models \alpha$.

Completeness: If $\Sigma \models \alpha$, then $\Sigma \vdash \alpha$.

Proof outline for soundness:

Each application of a rule is sound.

By induction, any finite # of rule applications is sound.

Soundness of $\forall e$ Oct 30

The $\forall e$ rule: $(\forall x\ \alpha) \vdash \alpha[t/x]$

for any formula $\alpha$, variable $x$, and term $t$.

Theorem: The $\forall e$ rule is sound. (Show that the entailment $(\forall x\ \alpha) \models \alpha[t/x]$ holds.)

Proof: - Let $\alpha$ be any predicate formula. Let $x$ be any variable. Let $t$ be any predicate term.

We need to show that $\{(\forall x\ \alpha)\} \models \alpha[t/x]$

- Consider an interpretation and environment $(I, E)$

Assume $I \models_E (\forall x\ \alpha)$. We need to show $\alpha[t/x]^{(I,E)} = T$.

- By definition of a $\forall$ formula, $I \models_E (\forall x\ \alpha)$ means that $\alpha^{(I, E[x \mapsto d])} = T$ for every $d \in dom(I)$. ①

- $t$ is a term. Thus $t^{(I,E)} \in dom(I)$. by Lemma 1

- By ①, $\alpha^{(I, E[x \mapsto t^{(I,E)}])} = T$

- Therefore, $\alpha[t/x]^{(I,E)} = T$. by Lemma 2.

QED.

Lemma 1: any predicate term evaluates to an element of the domain of an interpretation

$t^{(I,E)} \in dom(I)$ for any predicate term $t$, any interpretation $I$, and any environment $E$

Lemma 2: $\alpha[t/x]^{(I,E)} = \alpha^{(I, E[x \mapsto t^{(I,E)}])}$

for any predicate formula $\alpha$, any term $t$, any interpretation $I$, and any environment $E$.

Soundness of $\exists i$. Oct 30.

The $\exists i$ rule: $\alpha[t/x] \vdash (\exists x\ \alpha)$

for any formula $\alpha$, variable $x$, and term $t$.

Theorem: The $\exists i$ rule is sound.

Proof: Let $\alpha$ be any predicate formula. Let $x$ be any variable. Let $t$ be any term. We need to prove that $\alpha[t/x] \models (\exists x\ \alpha)$

Consider an interpretation and environment $(I, E)$.
Assume $I \models_E \alpha[t/x]$. We need to show $(\exists x\ \alpha)^{(I,E)} = T$.

$I \models_E \alpha[t/x]$ is the same as $\alpha[t/x]^{(I,E)} = T$.

By Lemma 2, $\alpha^{(I, E[x \mapsto t^{(I,E)}])} = T$. ①

By Lemma 1, $t^{(I,E)} \in dom(I)$.

By ①, $\alpha^{(I, E[x \mapsto d])} = T$ for $d = t^{(I,E)}$ which is a domain element

By definition of a $\exists$ formula, $I \models_E (\exists x\ \alpha)$.

QED

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Theorem (completeness): if $\Sigma \models \alpha$, then $\Sigma \vdash \alpha$.

Proof: We will prove the contrapositive of the theorem.

Assume $\Sigma \nvdash \alpha$. We need to prove that $\Sigma \not\models \alpha$.

Step 1: If $\Sigma \nvdash \alpha$, then $\Sigma \cup \{(\neg\alpha)\} \nvdash \alpha$.

Step 2: If $\Sigma \cup \{(\neg\alpha)\} \nvdash \alpha$, then there are $I$ and $E$ such that $I \models_E \Sigma \cup \{(\neg\alpha)\}$.

Step 3: If there are $I$ and $E$ such that $I \models_E \Sigma \cup \{(\neg\alpha)\}$, then $\Sigma \not\models \alpha$.

Proof of step 1: ~~We prove this by contradiction~~

~~By rule $\to i$, if $\Sigma \cup \{(\neg\alpha)\} \vdash \alpha$, then $\Sigma \vdash ((\neg\alpha) \to \alpha)$~~

~~If $\Sigma \vdash ((\neg\alpha) \to \alpha)$, then $\Sigma \vdash \alpha$, which~~

We prove this by contradiction.

Assume $\Sigma \nvdash \alpha$ and $\Sigma \cup \{(\neg\alpha)\} \vdash \alpha$.

By rule $\to i$, if $\Sigma \cup \{(\neg\alpha)\} \vdash \alpha$, then $\Sigma \vdash ((\neg\alpha) \to \alpha)$.

If $\Sigma \vdash ((\neg\alpha) \to \alpha)$, then $\Sigma \vdash \alpha$. because

| | | |
|---|---|---|
| 1. | $((\neg\alpha) \to \alpha)$ | premise |
| 2 | $(\neg\alpha)$ | assumption |
| 3 | $\alpha$ | $\to e$: 1, 2 |
| 4 | $\perp$ | $\perp i$: 2, 3 |
| 5 | $(\neg(\neg\alpha))$ | $\neg i$: 2–4 |
| 6 | $\alpha$ | $\neg\neg e$: 5. |

However, $\Sigma \vdash \alpha$ contradicts with our assumption $\Sigma \nvdash \alpha$

QED

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Proof of the completeness theorem continued:

Proof of step 3:

If there are $I$ and $E$ such that $I \models_E \Sigma \cup \{(\neg\alpha)\}$,

then $I$ and $E$ satisfy $\Sigma$ but do not satisfy $\alpha$.

By definition of entailment, this means that $\Sigma \not\models \alpha$

QED.

Proof sketch of step 2:

If $\Sigma \cup \{(\neg\alpha)\} \not\vdash \alpha$, then there are $I$ and $E$ such that

$I \models_E \Sigma \cup \{(\neg\alpha)\}$.

We need to construct $(I, E)$ that satisfy $\Sigma \cup \{(\neg\alpha)\}$.

Let dom$(I)$ contain all the terms.

Constants, variables, and functions all correspond to terms in dom$(I)$.

For each predicate, consider each formula in $\Sigma$ containing the predicate, we can find an interpretation that makes the formula true because $\Sigma \cup \{(\neg\alpha)\} \not\vdash \alpha$ means that the set of formulas $\Sigma \cup \{(\neg\alpha)\}$ does not lead to a contradiction.

QED