EQtrans(k). $\dfrac{t_1 = t_2 \quad t_2 = t_3 \quad \cdots \quad t_k = t_{k+1}}{t_1 = t_{k+1}}$ for any terms $t_1, \ldots, t_{k+1}$.

"applying transitivity k times".

Nov 6

Peano Arithmetic

EQsubs(r): $\dfrac{t_1 = t_2}{\underbrace{r[t_1/z]}_{\in D} \overset{*}{=} \underbrace{r[t_2/z]}_{\in D}}$ for any variable z, and terms r, t_1 and t_2

Axioms for Equality

EQ 1: $(\forall x\ (x = x))$ (reflexivity)

EQ 2: For each formula α and variable z,

$$(\forall x (\forall y\ ((x = y) \rightarrow (\underbrace{\alpha[x/z]}_{=T} \overset{*}{\rightarrow} \underbrace{\alpha[y/z]}_{=T}))))$$

EQsymm: $(\forall x (\forall y\ ((x = y) \rightarrow (y = x))))$ (symmetry)

EQtrans: $(\forall w\ (\forall x\ (\forall y\ ((x = y) \rightarrow ((y = w) \rightarrow (x = w))))))$ (transitivity)

Peano Arithmetic:

Constant symbols: 0
Function symbols: $s^{(1)}$, $+^{(2)}$, $\times^{(2)}$

Interpretation I:
Domain: $\mathbb{N}$ the set of natural numbers $\{0, 1, 2, \ldots\}$
0^I = zero.
$s^{(1)}$: the successor function. $s(x) = x + 1$, $\forall x \in \mathbb{N}$.
$+^{(2)}$: the addition function.
$\times^{(2)}$: the multiplication function.

Every natural number has a corresponding term:
0 : 0
1 : $s(0)$
2 : $s(s(0))$
3 : $s(s(s(0)))$

Peano Axioms: Nov 6

PA1: $(\forall x\ (\neg(s(x) = 0)))$

- Every natural number's successor is not zero.
- Zero is not a successor.

PA2: $(\forall x(\forall y.\ ((s(x) = s(y)) \rightarrow (x = y))))$

- If two numbers are the same, they must have the same predecessor. $P(x)$ $((x = y) \rightarrow (P(x) = P(y)))$.
- No number has two different predecessors.

Two properties of addition:

PA3: $(\forall x\ (x + 0 = x))$

- The addition of any number and zero is the same number.
- Adding zero to any number yields the same number

PA4: $(\forall x(\forall y.\ (x + s(y) = s(x + y))))$

- The addition of one number (x) and the successor of another (y). number is the successor of the sum of the two numbers
- Adding a successor yields the successor of adding the number.

Two properties of multiplication:

PA5: $(\forall x\ (x \times 0 = 0))$

- Multiplying any number by zero yields zero.

PA6: $(\forall x(\forall y.\ (x \times s(y) = (x \times y) + x)))$

- Multiplying one number and the successor of another number is the product of the two numbers plus the first number.

Peano Axioms. Nov. 6.

① Proving $(\forall x\ \varphi)$ by induction.

PA7: For each formula φ and variable x,

$$\Big(\underbrace{\varphi[0/x]}_{\text{base case}} \rightarrow \Big(\underbrace{\big(\forall x\,(\overbrace{\varphi[x/x]}^{\text{induction hypothesis}} \rightarrow \varphi[s(x)/x])\big)}_{\text{induction step}} \rightarrow \underbrace{(\forall x\ \varphi)}_{\text{conclusion}}\Big)\Big)$$

This axiom allows us to prove properties of natural numbers by induction.

Base case: We need to prove $\varphi[0/x]$.

Induction step: We need to prove

$$(\forall x\ (\varphi[x/x] \rightarrow \varphi[s(x)/x]))$$

Consider an arbitrary natural number k.
Induction hypothesis: assume $\varphi[k/x]$ is true
We need to prove $\varphi[s(k)/x]$ is true

...

By the principle of mathematical induction, $(\forall x\ \varphi)$ is true.

QED

② Proving $(\forall x\ \varphi)$ by $\forall i$ (direct proof).

Proof:

1	u fresh	assumption
⋮	⋮	
⋮	⋮	
n	$\varphi[u/x]$	
$n+1$	$(\forall x\ \varphi)$	$\forall i$: 1–n

Properties of Natural Numbers

Theorem: $(\forall x\, (\neg (S(x) = x)))$

Proof by induction:

Base case: We need to prove that $(\neg(S(0) = 0))$

1	$(\forall x\, (\neg(S(x) = 0)))$	PA1
2	$(\neg(S(0) = 0))$	$\forall e: 1\ [0/x]$

Induction step: We need to prove that

$\phi \vdash (\forall x\, ((\neg(S(x) = x)) \to (\neg(S(S(x)) = S(x)))))$

Hint: use $\neg i$ and PA2.

3	u fresh	assumption.
4	$(\neg(S(u) = u))$	assumption.
5	$(S(S(u)) = S(u))$	assumption.
6	$(\forall x\, (\forall y\, ((S(x) = S(y)) \to (x = y))))$	PA2
7	$(S(S(u)) = S(u)) \to (S(u) = u)$	
8	$(\forall y\, ((S(S(u)) = S(y)) \to (S(u) = y)))$	$\forall e: 6\ [S(u)/x]$
9	$((S(S(u)) = S(u)) \to (S(u) = u))$	$\forall e: 8\ [u/y]$
10		
11		
12		
13	$(S(u) = u)$	$\to e: 5, 9$
14	$\perp$	$\perp i: 4, 13$
15	$(\neg(S(S(u)) = S(u)))$	$\neg i: 5\text{–}14$
16	$((\neg(S(u) = u)) \to (\neg(S(S(u)) = S(u))))$	$\to i: 4\text{–}15$
17	$(\forall x\, ((\neg(S(x) = x)) \to (\neg(S(S(x)) = S(x)))))$	$\forall i: 3\text{–}16$.

Properties of Natural Numbers. Nov 6.

Theorem: $(\forall x\ (\neg (S(x) = x)))$.

Proof by induction continued:

Let's finish the proof.

15 $((\neg(S(0) = 0)) \to ((\forall x\ ((\neg(S(x) = x)) \to (\neg(S(S(x)) = S(x))))) \to (\forall x\ (\neg(S(x) = x)))))$ PA7: $[\varphi = (\neg(S(x) = x))]$

16 $((\forall x\ ((\neg(S(x) = x)) \to (\neg(S(S(x)) = S(x))))) \to (\forall x\ (\neg(S(x) = x))))$ $\to e$: 2, 15

17 $(\forall x (\neg(S(x) = x)))$ $\to e$: 14, 16.

QED

Properties of Natural Numbers. Nov 6

Theorem: $\emptyset \vdash (\forall x (\forall y \, (x + y = y + x)))$.
"Addition is commutative."

Proof by induction on x.

x Base case: We need to prove that

$(\forall y \, (0 + y = y + 0))$.

We prove this by induction on y.

y Base case: We need to prove that $(0 + 0 = 0 + 0)$.

1	$(\forall x \, (x = x))$.	EQ1.
2	$(0+0 = 0+0)$	$\forall e$: 1 $[(0+0)/x]$

y Induction step: We need to prove that

$(\forall y \, ((0 + y = y + 0) \rightarrow (0 + S(y) = S(y) + 0)))$.

Hint: use PA4, PA3 and EQsubs(r).

3	u fresh	assumption
4	$0 + u = u + 0$.	assumption.
5		
6		
7		
8		
9		
10		
11		
12		
13		
14		
15	$(0 + S(u) = S(u) + 0)$	
16	$((0 + u = u + 0) \rightarrow (0 + S(u) = S(u) + 0))$	$\rightarrow i$: 4–15
17	$(\forall y \, ((0 + y = y + 0) \rightarrow (0 + S(y) = S(y) + 0)))$	$\forall i$: 3–16

Proof by induction on x continued.

Base case: We need to prove that $(\forall y\,(0+y=y+0))$.

18 $((0+0=0+0) \to ((\forall y\,(0+y=y+0) \to (0+S(y)=S(y)+0)) \to (\forall y\,(0+y=y+0))))$ PA7.

19 $((\forall y\,(0+y=y+0) \to (0+S(y)=S(y)+0)) \to (\forall y\,(0+y=y+0)))$ $\to$e: 2, 18

20 $(\forall y\,(0+y=y+0))$ $\to$e: 17, 19.

This completes the proof of the base case.

(x) Induction step: We need to prove that
$(\forall x\ (\forall y\,(x+y=y+x)) \to (\forall y\,(S(x)+y=y+S(x))))$.

Consider an arbitrary $x \in \mathbb{N}$. We will prove that
$((\forall y\,(x+y=y+x)) \to (\forall y\,(S(x)+y=y+S(x))))$.

Assume that $(\forall y\,(x+y=y+x))$. ⊛
We will prove that $(\forall y\,(S(x)+y=y+S(x)))$.
We prove this by induction on y.

(y) Base case: We will prove that $S(x)+0=0+S(x)$.

21

22

23

24

25

Proof by induction on x continued:

Induction step continued:

Induction on y continued:

(y) Induction step: We will prove that

$(\forall y\, ((S(x)+y=y+S(x)) \rightarrow (S(x)+S(y)=S(y)+S(x))))$.

Choose a fresh variable u.

We will prove $((S(x)+u=u+S(x)) \rightarrow (S(x)+S(u)=S(u)+S(x)))$.

Assume $(S(x)+u=u+S(x))$. $(**)$

We will prove $(S(x)+S(u)=S(u)+S(x))$.

30 $S(x)+S(u)= S(S(x)+u)$

31 $S(S(x)+u)=S(u+S(x))$

32 $S(u+S(x))=S(S(u+x))$

33 $S(S(u+x))=S(S(x+u))$

34 $S(S(x+u))=S(x+S(u))$

35 $S(x+S(u))=S(S(u)+x)$

36 $S(S(u)+x)=S(u)+S(x)$

37 $S(x)+S(u)=S(u)+S(x)$