Natural Deduction Rules for Predicate Logic

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① $\forall$- elimination

$$\frac{(\forall x\ \alpha)}{\alpha[t/x]}\ \forall e \qquad\qquad \frac{(x \wedge y)}{x}\ \wedge e$$

② $\forall$- introduction

Box denotes the scope of var u.

$$\frac{\boxed{\begin{array}{c} u \text{ fresh} \\ \vdots \\ \alpha[u/x] \end{array}}}{(\forall x\ \alpha)}\ \forall i \qquad\qquad \frac{x \quad y}{(x \wedge y)}\ \wedge i$$

u fresh → chooses a fresh var

particular case → $\alpha[u/x]$

general case → $(\forall x\ \alpha)$

- know nothing about u except that u is an element of our domain. (If u is special, our conclusion may not be valid.)
- The fresh variable u: can not escape the box. e.g. cannot conclude $\alpha[u/x]$ outside of the box.
- When you choose u, make sure it does not appear anywhere ~~else in the proof, especially~~ outside of the box in the proof.

③ $\exists$- elimination

choose a fresh var

make assumption → $\alpha[u/x]$, u fresh

$$\frac{(\exists x\ \alpha) \quad \boxed{\begin{array}{c} \alpha[u/x],\ u \text{ fresh} \\ \vdots \\ \beta \end{array}}}{\beta}\ \exists e$$

(proof by cases)

$$\frac{(x \vee y) \quad \boxed{\begin{array}{c} x \\ \vdots \\ \beta \end{array}} \quad \boxed{\begin{array}{c} y \\ \vdots \\ \beta \end{array}}}{\beta}\ \vee e$$

conclusion β may have nothing to do with the starting formula $(\exists x\ P(x))$ or $(x \vee y)$.

④ $\exists$- introduction

$$\frac{\alpha[t/x]}{(\exists x\ \alpha)}\ \exists i \qquad\qquad \frac{x}{(x \vee y)}\ \vee i$$

solutions

Natural Deduction Examples.

$(\forall x\ \overset{\alpha}{\overbrace{P(x)}})$ $t=u$.

$\alpha[u/x] = P(u)$.

$\forall e$? ② which one do I want?

△① $\{(\forall x\ P(x))\} \vdash (\exists y\ P(y))$.

Proof:

1	$(\forall x\ P(x))$	premise
2	$P(u)$	$\forall e$: 1
3	$(\exists y\ P(y))$	$\exists i$: 2

$\exists i$? ① $P(u) \longrightarrow \alpha[u/y]$.

$(\exists y\ \underbrace{P(y)}_{\alpha})$ $t=u$.

△② $\{(\forall x\ P(x))\} \vdash (\forall y\ P(y))$

Proof:

1	$(\forall x\ P(x))$	premise
2	u fresh	
3	$P(u)$	$\forall e$: 1
4	$(\forall y\ P(y))$	$\forall i$: 2–3

$\forall e$? ②

$(\forall x\ P(x))$

u fresh

⋮

$P(u)$.

$\forall i$? ①

u fresh

⋮

$P(u)$.

$(\forall y\ P(y))$

△③ $\{(\exists x\ R(x))\} \vdash (\exists y\ R(y))$.

Proof:

1	$(\exists x\ R(x))$	premise
2	$R(u)$, u fresh.	assumption
3	$(\exists y\ R(y))$	$\exists i$: 2
4	$(\exists y\ R(y))$	$\exists e$: 1, 2–3

$\exists e$? ①

$(\exists x\ R(x))$

$R(u)$ u fresh

⋮

$(\exists y\ R(y))$

$\exists i$? ②

$(\exists y\ R(u))$

$\{(\exists y\ (\forall x\ P(x,y)))\} \vdash (\forall x\ (\exists y\ P(x,y)))$.

Proof:

1.	$(\exists y\ (\forall x\ P(x,y)))$	premise
2	$(\forall x\ P(x,w))$, w fresh	assumption.
3	u fresh	
4	$P(u,w)$	$\forall e$: 2
5	$(\exists y\ P(u,y))$	$\exists i$: 4
6	$(\forall x\ (\exists y\ P(x,y)))$	$\forall i$: 3–5
7	$(\forall x\ (\exists y\ P(x,y)))$	$\exists e$: 1, 2–6.

Natural Deduction Examples

① $\{P(t), (\forall x (P(x) \to (\neg Q(x))))\} \vdash (\neg Q(t))$

Proof:

1	$P(t)$	premise
2	$(\forall x (P(x) \to (\neg Q(x))))$	premise ①
3	$(P(t) \to (\neg Q(t)))$	$\forall e$: 2 ③
4	$(\neg Q(t))$	$\to e$: 1, 3 ②

② $\{(\neg P(y))\} \vdash (\exists x (P(x) \to Q(y)))$

Proof:

1	$(\neg P(y))$	premise
2	$P(y)$	assumption
3	$\perp$	$\perp i$: 1, 2
4	$Q(y)$	$\perp e$: 3
5	$(P(y) \to Q(y))$	$\to i$: 2–4
6	$(\exists x (P(x) \to Q(y)))$	$\exists i$: 5 ①

③ fill in the subproof.

② $\to i$? set up the structure

~~premise has P(y) in it~~

①

Natural Deduction Examples

① $\{(\forall x\,(P(x) \to Q(x))),\ (\forall x\,P(x))\} \vdash (\forall x\,Q(x))$.

Proof:

1	$(\forall x\,(P(x) \to Q(x)))$	premise	$\forall e$? which one?
2	$(\forall x\,P(x))$	premise.	$\forall e$?
3	u fresh		
4	$P(u)$	$\forall e$: 2	②
5	$(P(u) \to Q(u))$	$\forall e$: 1.	
6	$Q(u)$	$\to e$: 4,5	
7	$(\forall x\,Q(x))$	$\forall i$	$\forall i$? ①.

☆ ② $\{(\forall x\,(P(x) \to Q(x)))\} \vdash ((\forall x\,P(x)) \to (\forall y\,Q(y)))$.

Proof:

1	$(\forall x\,(P(x) \to Q(x)))$	premise	$\forall e$?
2	$(\forall x\,P(x))$	assumption.	$\forall e$?
3	u fresh		
4	$P(u)$	$\forall e$: 2	③
5	$(P(u) \to Q(u))$	$\forall e$: 1	
6	$Q(u)$	$\to e$: 4,5	
7	$(\forall y\,Q(y))$	$\forall i$: 3-6	$\forall i$? ②
8	$((\forall x\,P(x)) \to (\forall y\,Q(y)))$	$\to i$: 2-7	$\to i$? ①

The following proof doesn't work.

1	$(\forall x\,(P(x) \to Q(x)))$	premise.
2	$(\forall x\,P(x))$	assumption.
3	$P(w)$	$\forall e$:
4	$P(w) \to Q(w)$	$\forall e$:
5	$Q(w)$	$\to e$:
6	u fresh	
	$Q(u)$ $Q(w)$.	
7	$(\forall y\,Q(y))$	
8	$((\forall x\,P(x)) \to (\forall y\,Q(y)))$.	$\to i$: 2-7

Natural Deduction Examples

① $\{(\forall x\,(P(x) \rightarrow Q(x))),\ (\exists x\ P(x))\} \vdash (\exists x\ Q(x))$

Proof:

1	$(\forall x\,(P(x) \rightarrow Q(x)))$	premise	$\forall e$? ②
2	$(\exists x\ P(x))$	premise	$\exists e$? ①
3	$P(u)$, u fresh	assumption	
4	$P(u) \rightarrow Q(u)$	$\forall e$: 1.	
5	$Q(u)$	$\rightarrow e$: 3,4.	
6	$(\exists x\ Q(x))$	$\exists i$: 5	$\exists i$?
7	$(\exists x\ Q(x))$.	$\exists e$: 3-6	$\exists i$?

② $\{(\forall x\,(Q(x) \rightarrow R(x))),\ (\exists x\,(P(x) \land Q(x)))\} \vdash (\exists x\,(P(x) \land R(x)))$

Proof:

1	$(\forall x\ (Q(x) \rightarrow R(x)))$	premise	$\forall e$?
2	$(\exists x\ (P(x) \land Q(x)))$	premise	$\exists e$? ①
3	$(P(u) \land Q(u))$, u fresh	assumption.	
4	$P(u)$	$\land e$: 3	
5	$Q(u)$	$\land e$: 3	
6	$Q(u) \rightarrow R(u)$	$\forall e$: 1.	
7	$R(u)$	$\rightarrow e$: 5,6	
8	$P(u) \land R(u)$	$\land i$: 4,7	
9	$(\exists x\,(P(x) \land R(x)))$	$\exists i$: 8	
10	$(\exists x\,(P(x) \land R(x)))$	$\exists e$:	$\exists i$?

△ ③ $\{(\exists x\ P(x)),\ (\forall x\,(\forall y\ (P(x) \rightarrow Q(y))))\} \vdash (\forall y\ Q(y))$

Proof:

1	$(\exists x\ P(x))$	premise	$\exists e$? ②
2	$(\forall x\,(\forall y\ (P(x) \rightarrow Q(y))))$	premise.	$\forall e$? ③, ④
3	u fresh		
4	$P(w)$, w fresh	assumption.	
5	$(\forall y\ (P(w) \rightarrow Q(y)))$	$\forall e$: 2. ③	
6	$P(w) \rightarrow Q(u)$	$\forall e$: 5 ④	
7	$Q(u)$	$\rightarrow e$: 4,6	
8	$Q(u)$	$\exists e$: 1, 4-7	
9	$(\forall y\ Q(y))$	$\forall i$: 3-8	$\forall i$? ①

Examples:

△ ⑪ $\{(\exists x\,(\neg P(x)))\} \vdash (\neg(\forall x\; P(x)))$ De Morgan's ②

Proof:

1	$(\exists x\; (\neg P(x)))$	premise
2	$(\forall x\; P(x))$	assumption
3	$(\neg P(u))$; u fresh	assumption
4	$P(u)$	$\forall e$: 2
5	$\perp$	$\perp i$: 3,4
6	$\perp$	$\exists e$: 1, 3–5
7	$(\neg(\forall x\; P(x)))$	$\neg i$: 2–6

Proof:

1	$(\exists x\; (\neg P(x)))$	premise
2	$(\neg P(u))$, u fresh	assumption
3	$(\forall x\; P(x))$	assumption
4	$P(u)$	$\forall e$: 3
5	$\perp$	$\perp i$: 2,4
6	$(\neg(\forall x\; P(x)))$	$\neg i$: 3–5
7	$(\neg(\forall x\; P(x)))$	$\exists e$: 1, 2–6

⑫ $\{(\neg(\exists x\; P(x)))\} \vdash (\forall x\; (\neg P(x)))$ De Morgan's ①

Proof:

1	$(\neg(\exists x\; P(x)))$	premise
2	u fresh	
3	$P(u)$	assumption
4	$(\exists x\; P(x))$	$\exists i$: 3
5	$\perp$	$\perp i$: 1,4
6	$(\neg P(u))$	$\neg i$: 3–5
7	$(\forall x\; (\neg P(x)))$	$\forall i$: 2–6

Examples:

⑧ $\{(\forall x(\neg P(x)))\} \vdash (\neg(\exists x\, P(x)))$. De Morgan's ③

Proof:

1	$(\forall x\,(\neg P(x)))$	premise
2	$(\exists x\, P(x))$	assumption
3	$P(u)$, u fresh	assumption
4	$(\neg P(u))$	$\forall e$: 1.
5	$\bot$	$\bot i$: 3,4
6	$\bot$	$\exists e$: 2, 3–5
7	$(\neg(\exists x\, P(x)))$	$\neg i$: 2–6

⑨ $\{(\forall x\;(P(x) \to Q(x)))\} \vdash ((\forall x\; P(x)) \to (\forall y\; Q(y)))$

Proof: $\forall e$, $\forall i$

1	$(\forall x\;(P(x) \to Q(x)))$	premise
2	$(\forall x\; P(x))$	assumption
3	u fresh	
4	$P(u)$	$\forall e$: 2
5	$P(u) \to Q(u)$	$\forall e$: 1
6	$Q(u)$	$\to e$:
7	$(\forall y\; Q(y))$	$\forall i$: 3–6
8	$((\forall x\; P(x)) \to (\forall y\; Q(y)))$.	$\to i$: 2–7

⑩ $\{(\exists x\; R(x))\} \vdash (\exists y\; R(y))$

$\exists i$, $\exists e$.

Proof:

1	$(\exists x\; R(x))$	premise
2	$R(u)$, u fresh	assumption
3	$(\exists y\, R(y))$	$\exists i$: 2
4	$(\exists y\; R(y))$	$\exists e$: 1, 2–3

Examples :

⑬ $\{(\neg(\forall x\ P(x)))\} \vdash (\exists x\ (\neg P(x)))$ De Morgan's ④

Proof:

1	$(\neg(\forall x\ P(x)))$	premise
2	$(\neg(\exists x\ (\neg P(x))))$	assumption
3	u fresh	
4	$(\neg P(u))$	assumption
5	$(\exists x\ (\neg P(x)))$	$\exists i$: 4.
6	$\perp$	$\perp i$: 2,5.
7	$(\neg(\neg P(u)))$	$\neg i$: 4–6
8	$P(u)$	$\neg\neg e$: 7
9	$(\forall x\ P(x))$	$\forall i$: 3–8
10	$\perp$	$\perp i$: 1, 9
11	$(\neg(\neg(\exists x(\neg P(x)))))$	$\neg i$: 2–10
12	$(\exists x(\neg P(x)))$	$\neg\neg e$: 11

⑭ $\{(\neg(p \wedge q))\} \vdash ((\neg p) \vee (\neg q))$

Proof:

1	$(\neg(p \wedge q))$	premise
2	$(\neg((\neg p) \vee (\neg q)))$	assumption.
3	$(\neg p)$	assumption
4	$((\neg p) \vee (\neg q))$	$\vee i$: 3
5	$\perp$	$\perp i$: 2,4
6	$(\neg(\neg p))$	$\neg i$: 3–5
7	p	$\neg\neg e$: 6
8	$(\neg q)$	assumption
9	$((\neg p) \vee (\neg q))$	$\vee i$: 8
10	$\perp$	$\perp i$: 2,9
11	$(\neg(\neg q))$	$\neg i$: 8–10
12	q	$\neg\neg e$: 11
13	$(p \wedge q)$	$\wedge i$: 7,12
14	$\perp$	$\perp i$: 1, 13
15	$(\neg(\neg((\neg p) \vee (\neg q))))$	$\neg i$: 2–14
16	$((\neg p) \vee (\neg q))$	$\neg\neg e$: 15