Axioms of equality. Nov 1.

- An axiom is a formula that can be used as a premise in a natural deduction proof.
  - Think of axioms as additional inference rules that you can use in a natural deduction proof.

- Predicate logic with equality adds the restriction that the symbol "=" must be interpreted as equality on the domain.

$$(=)^I = \{<d,d> \mid d \in dom(I)\}$$

Axioms for equality.

EQ1: $(\forall x \ (x = x))$

EQ2: For each formula α and variable z,

$$(\forall x \ (\forall y \ ((x = y) \rightarrow (\alpha[x/z] \rightarrow \alpha[y/z]))))$$

If x and y are the same and taking α and replacing every z in α by x leads to a true formula, then taking α and replacing every z in α by y also leads to a true formula.

Using these axioms, we can prove two properties of $=$.

① symmetry

$$\vdash (\forall x \ (\forall y \ ((x = y) \rightarrow (y = x))))$$

② transitivity

$$\vdash (\forall w \ (\forall x \ (\forall y \ ((x = y) \rightarrow ((y = w) \rightarrow (x = w))))))$$

Proof of the Symmetry of "="

Nov 1.

Show that $\vdash (\forall x (\forall y ((x=y) \rightarrow (y=x))))$

Proof:

1	a fresh	assumption
2	b fresh	assumption
3	$(a=b)$	assumption
4	$\forall x \forall y ((x=y) \rightarrow ((x=a) \rightarrow (y=a)))$	EQ2 $[z=a]$
5	$\forall y ((a=y) \rightarrow ((a=a) \rightarrow (y=a)))$	$\forall e$: 4
6	$((a=b) \rightarrow ((a=a) \rightarrow (b=a)))$	$\forall e$: 5
7	$((a=a) \rightarrow (b=a))$	$\rightarrow e$: 3,6
8	$\forall x (x=x)$	EQ1
9	$(a=a)$	$\forall e$: 8
10	$(b=a)$	$\rightarrow e$: 7,9
11	$((a=b) \rightarrow (b=a))$	$\rightarrow i$: 3–10
12	$(\forall y ((a=y) \rightarrow (y=a)))$	$\forall i$: 2–11
13	$(\forall x (\forall y ((x=y) \rightarrow (y=x))))$	$\forall i$: 1–12

We know $(a=b)$ $\quad x = z \Rightarrow a$, z is a $\quad y = z \Rightarrow a$, y is b

$\forall x \forall y ((x=y) \rightarrow (\alpha[x/z] \rightarrow \alpha[y/z]))$

$a=b \quad \underline{a=a} \quad \underline{b=a}$

$((a=b) \rightarrow ((a=a) \rightarrow (b=a)))$

$\forall x (x=x)$

$a=a.$

Our goal $(b=a)$.

Proof of the Transitivity of "="

Nov 1.

Show that $\vdash (\forall w (\forall x (\forall y ((x=y) \rightarrow ((y=w) \rightarrow (x=w))))))$.

Proof:

1	a fresh	
2	b fresh	
3	c fresh	
4	$(b=c)$	assumption
5	$\forall x \forall y ((x=y) \rightarrow ((x=a) \rightarrow (y=a)))$	EQ2 [z=a]
6	$\forall y ((c=y) \rightarrow ((c=a) \rightarrow (y=a)))$	$\forall$e: 5
7	$((c=b) \rightarrow ((c=a) \rightarrow (b=a)))$	$\forall$e: 6.
8	$\forall x \forall y ((x=y) \rightarrow (y=x))$	EQ symmetry.
9	$\forall y ((b=y) \rightarrow (y=b))$	$\forall$e: 8
10	$((b=c) \rightarrow (c=b))$.	$\forall$e: 9
11	$(c=b)$	$\rightarrow$e: 4, 10.
12	$((c=a) \rightarrow (b=a))$	$\rightarrow$e: 7, 11
13	$((b=c) \rightarrow ((c=a) \rightarrow (b=a)))$	$\rightarrow$i: 4–12
14	$\forall y ((b=y) \rightarrow ((y=a) \rightarrow (b=a)))$	$\forall$i: 3–13
15	$\forall x \forall y ((x=y) \rightarrow ((y=a) \rightarrow (x=a)))$	$\forall$i: 2–14.
16	$\forall w \forall x \forall y ((x=y) \rightarrow ((y=w) \rightarrow (x=w)))$.	$\forall$i: 1–15.

We know $(b=c)$

$$\forall x \forall y ((x=y) \rightarrow (\alpha[x/z] \rightarrow \alpha[y/z]))$$

$$c=b \qquad c=a \qquad b=a.$$

$$((c=b) \rightarrow ((c=a) \rightarrow (b=a)))$$

Our goal $((c=a) \rightarrow (b=a))$