Oct 31.

Proving / Disproving the soundness of an inference rule.

Q1: Prove that the $\vee i$ rule is sound.

$$\frac{x}{(x \vee y)} \vee i$$

(Show that the entailment $\{x\} \models (x \vee y)$ holds.)

Proof: Let t be a truth valuation. Assume $x^t = T$.
We need to prove that $(x \vee y)^t = T$.

By the definition of $\vee$, if $x^t = T$, then $(x \vee y)^t = T$.

QED

Q2: Prove that the following $\vee e^*$ rule is not sound.

$$\frac{(x \vee y)}{x} \vee e^*$$

(Give propositional formulas x and y such that $\{(x \vee y)\} \not\models x$.)

Proof: Let x be p and y be q where p and q are propositional variables.

Consider a truth valuation t such that $p^t = F$ and $q^t = T$.

We have that $x^t = p^t = F$ and $(x \vee y)^t = (p \vee q)^t = T$ by the definition of $\vee$.

Therefore, $\{(x \vee y)\} \not\models x$.

QED

☆ Alice: I forgot to choose specific formulas for x and y for the 8:30 am section. This is necessary for the proof. Otherwise, if $x = (a \vee (\neg a))$, it is not possible to find a truth valuation t such that $x^t = F$

Proving Semantic Entailment.

Q: Show that $(\exists y (\forall x\ P(x,y))) \models (\forall x (\exists y\ P(x,y)))$.

Proof: Consider an interpretation I.

Assume that $(\exists y (\forall x\ P(x,y)))^I = T$.

We need to prove that $(\forall x (\exists y\ P(x,y)))^I = T$.

- $(\exists y (\forall x\ P(x,y)))^I = T$ holds.

By def of a $\exists$ formula, for any environment E,

$$(\forall x\ P(x,y))^{(I, E[y \mapsto a])} = T \text{ for some } a \in dom(I).$$

By def of a $\forall$ formula,

$$P(x,y)^{(I, E[y \mapsto a][x \mapsto d])} = T \text{ for some } \underline{a} \in dom(I) \text{ and every } d \in dom(I).$$

☆ We chose the value a for y first. Thus, to make P(x, y) true, the value of a for y does not depend on the value for x. Therefore, we add back the two quantifiers in reverse order and the resulting formula is still true.

By def of a $\exists$ formula,

$$(\exists y\ P(x,y))^{(I, E[x \mapsto d])} = T \text{ for every } d \in dom(I).$$

By def of a $\forall$ formula,

$$(\forall x (\exists y\ P(x,y)))^{(I,E)} = T.$$

QED.