Lemma 2: Every well-formed formula has an equal number of opening and closing brackets. (This is proved in a separate handout. For this proof, we assume that this lemma is true.)

Lemma 3: Every proper prefix of a well-formed formula \( \varphi \) has more opening than closing brackets.

Define \( P(\varphi) \) to be that every proper prefix of \( \varphi \) has more opening than closing brackets.

Proof by structural induction:

**Base case:** \( \varphi \) is a propositional variable. We need to prove that \( P(\varphi) \) holds.

A propositional variable has no proper prefix. Thus, the theorem is true and \( P(\varphi) \) holds.

**Induction step:**

Let \( op(x) \) and \( cl(x) \) denote the number of opening and closing brackets in \( x \) respectively.

Case 1: \( \varphi \) is a well-formed formula of the form \( (\neg x) \) where \( x \) is a well-formed formula.

Induction hypothesis: Assume that \( P(x) \) holds. Let \( m \) denote any proper prefix of \( x \). The induction hypothesis becomes that \( m \) has more opening than closing brackets. We need to prove that \( P((\neg x)) \) holds.

There are four possible proper prefixes of \( (\neg x) \): \( (), (\neg), (\neg m), \) and \( (\neg x) \). We’ll prove the four cases separately below.

Case a: \( op(()) = 1 \) by inspection. \( cl(()) = 0 \) by inspection. \( op(()) > cl(()) \)

Case b: \( op((\neg)) = 1 \) by inspection. \( cl((\neg)) = 0 \) by inspection. \( op((\neg)) > cl((\neg)) \)

Case c: \( op((\neg m)) = 1 + op(m) \) by inspection of \( (\neg m) \)

\[ > 1 + cl(m) \] by the induction hypothesis

\[ > cl(m) \] algebra

\[ = cl((\neg m)) \] by inspection of \( (\neg m) \)

Case d:

\[ op((\neg x)) = 1 + op(x) \] by inspection \( (\neg x) \)

\[ = 1 + cl(x) \] by Lemma 2 and \( x \) is a well-formed formula

\[ > cl(x) \] algebra

\[ = cl((\neg x)) \] by inspection \( (\neg x) \)

Thus, \( P((\neg x)) \) holds.

(Proof continued on the next page)
Case 2: \( \varphi \) is a well-formed formula of the form \((x * y)\) where \(x\) and \(y\) are well-formed formulas and * is one of the four binary connectives \((\land, \lor, \to, \leftrightarrow)\).

Induction hypothesis: Assume that \(P(x)\) and \(P(y)\) hold. Let \(m\) and \(n\) denote any proper prefix of \(x\) and \(y\) respectively. The induction hypothesis becomes that each of \(m\) and \(n\) has more opening than closing brackets.

We need to prove that \(P((x * y))\) holds.

There are six possible proper prefixes of \((x * y)\): \((), (m, (x, (x * (x * n), (x * y). We’ll prove the six cases separately below.

Case a: \(o_p(()) = 1\) by inspection. \(c_l(()) = 0\) by inspection. \(o_p(()) > c_l(())\)

Case b: \(o_p(m)\)

\[
= 1 + o_p(m) \quad \text{by inspection of } (m) \\
> 1 + c_l(m) \quad \text{by the induction hypothesis} \\
> c_l(m) \quad \text{algebra} \\
= c_l(m) \quad \text{by inspection of } (m)
\]

Case c: \(o_p(x)\)

\[
= 1 + o_p(x) \quad \text{by inspection of } (x) \\
= 1 + c_l(x) \quad \text{by Lemma 2 and } x \text{ is a well-formed formula} \\
> c_l(x) \quad \text{algebra} \\
= c_l(x) \quad \text{by inspection of } (x)
\]

Case d: \(o_p(x * )\)

\[
= 1 + o_p(x) \quad \text{by inspection of } (x * ) \\
= 1 + c_l(x) \quad \text{by Lemma 2 and } x \text{ is a well-formed formula} \\
> c_l(x) \quad \text{algebra} \\
= c_l(x) \quad \text{by inspection of } (x * )
\]

Case e: \(o_p(x * n)\)

\[
= 1 + o_p(x) + o_p(n) \quad \text{by inspection } (x * n) \\
= 1 + c_l(x) + o_p(n) \quad \text{by Lemma 2 and } x \text{ is a well-formed formula} \\
> 1 + c_l(x) + c_l(n) \quad \text{by the induction hypothesis} \\
> c_l(x) + c_l(n) \quad \text{algebra} \\
= c_l(x * n) \quad \text{by inspection } (x * n)
\]

Case f: \(o_p(x * y)\)

\[
= 1 + o_p(x) + o_p(y) \quad \text{by inspection } (x * y) \\
= 1 + c_l(x) + c_l(y) \quad \text{by Lemma 2 and } x \text{ and } y \text{ are well-formed formulas} \\
> c_l(x) + c_l(y) \quad \text{algebra} \\
= c_l(x * y) \quad \text{by inspection } (x * y)
\]

By the principle of structural induction, \(P(\varphi)\) holds for every well-formed formula \(\varphi\). QED