Lemma 1: Every well-formed formula starts with a propositional variable or an opening bracket.

Lemma 2: Every well-formed formula has an equal number of opening and closing brackets.

Lemma 3: Every proper prefix of a well-formed formula has more opening brackets than closing brackets.

**Theorem: There is a unique way to construct every well-formed formula.**

Proof by structural induction:

Let $x$ be a well-formed formula. We want to prove that there is a unique way to construct $x$ as a well-formed formula.

Base case: $x$ is a propositional variable.

**Can we construct $x$ as ($\neg a$) for a well-formed formula $a$ by applying negation as the last step?**

If we construct a formula by applying negation as the last step, then it has to be of the form ($\neg a$) and has to contain at least 3 symbols. However, the formula $x$ only has 1 symbol. Therefore, we cannot construct $x$ by applying negation as the last step.

**Can we construct $x$ as ($a*b$) for well-formed formulas $a$ and $b$ by applying a binary connective * as the last step?**

If we construct a formula by applying negation as the last step, then it has to be of the form ($\neg a$) and has to contain at least 3 symbols. However, the formula $x$ only has 1 symbol. Therefore, we cannot construct $x$ by applying negation as the last step.
Induction step:

Case 1: x is (¬a) for a well-formed formula a.

Induction hypothesis: assume that there is a unique way to construct a. We need to prove that there is a unique way to construct (¬a).

We already know one way to construct (¬a): construct a, and apply negation as the last step. We need to show that there is no other way to construct (¬a).

Can we construct (¬a) as a propositional variable?

If we construct a formula as a propositional variable, then it has to have 1 symbol. However, the formula (¬a) has at least 3 symbols. So we cannot construct (¬a) as a propositional variable.

Can we construct (¬a) as (c*d) for well-formed formulas c and d by applying a binary connective * as the last step?

Suppose that we can construct (¬a) as (c*d) for well-formed formulas c and d by applying a binary connective * as the last step. Then the connective * has to be in the formula a. Let a = m*n. Then c = ¬m and d = n. We will argue that c is not a well-formed formula.

m is a proper prefix of the well-formed formula a. By Lemma 3, m has more opening than closing brackets. Thus, c also has more opening than closing brackets. By Lemma 2, c is not a well-formed formula.

Therefore, we cannot construct (¬a) by applying a binary connective as the last step.
Case 2: \( x \) is \((a*b)\) for well-formed formulas \(a\) and \(b\) where * is one of \(\land, \lor, \rightarrow, \) and \(\leftrightarrow\).

Induction hypothesis: assume that there is a unique way to construct \(a\) and \(b\) respectively. We need to prove that there is a unique way to construct \((a*b)\). We already know one way to construct \((a*b)\): construct \(a\) and \(b\) separately, and apply * as the last step. We need to show that there is no other way to construct \((a*b)\).

Can \((a*b)\) be constructed as a propositional variable?

If we construct a formula as a propositional variable, then it has to have 1 symbol. However, the formula \((a*b)\) has at least 3 symbols. So we cannot construct \((a*b)\) as a propositional variable.

Can \((a*b)\) be constructed as \((\neg c)\) for well-formed formula \(c\) by applying negation as the last step?

Suppose that we can construct \((a*b)\) by applying negation as the last step. Then the binary connective * has to be in \(c\). Let \(c = m*n\). Then \(a = \neg m\) and \(b = n\). We will argue that \(a\) is not a well-formed formula.

\(m\) is a proper prefix of the well-formed formula \(c\). By Lemma 3, \(m\) has more opening than closing brackets. Thus, \(a\) also has more opening than closing brackets. By Lemma 2, \(a\) is not a well-formed formula.
Can \((a*b)\) be constructed as \((c@d)\) for well-formed formulas \(c\) and \(d\) by applying a binary connective \(\@\) that is different from \(*\) as the last step?

Suppose that we can construct \((a*b)\) by applying a different binary connective \(\@\) as the last step. Then the binary connective \(\@\) has to be either in \(a\) or in \(b\).

If the binary connective \(\@\) is in \(a\), then \(c\) is a proper prefix of \(a\). By Lemma 3, \(c\) has more opening than closing brackets. Thus, \(c\) is not a well-formed formula.

If the binary connective \(\@\) is in \(b\), then let \(b = m@n\). Then \(c = a*m\) and \(d = n\). Let \(\text{op}(x)\) and \(\text{cl}(x)\) denote the number of opening and closing brackets in a formula \(x\).

\(a\) is a well-formed formula, so \(\text{op}(a) = \text{cl}(a)\) by Lemma 2.
\(m\) is a proper prefix of the well-formed formula \(b\), so \(\text{op}(m) > \text{cl}(m)\). By inspection of \(c\), \(\text{op}(c) = \text{op}(a) + \text{op}(m) > \text{cl}(a) + \text{cl}(m) = \text{cl}(c)\).
Thus, \(c\) has more opening than closing brackets. By Lemma 3, \(c\) is not a well-formed formula.

QED