Pathwidth and small-height drawings of 2-connected outer-planar graphs

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Abstract

In this paper, we study planar drawings of 2-connected outer-planar graphs. In an earlier paper we showed that every such graph has a visibility representation with height $O(\log n)$. In this paper, we show that with a different construction, the height is $4pw(G) - 3$, where $pw(G)$ denotes the pathwidth of graph G. Since for any planar graph G, any planar drawing has height $\geq pw(G)$, this is a 4-approximation algorithm for the height. We also show that our visibility representations can be converted into straight-line drawings of the same height.

Keywords: Outer-planar graph, pathwidth, graph drawing, approximation algorithm.

1 Introduction

Graph drawing is the art of creating a pretty picture of a graph. Since "pretty" is hard to define, common measures used are to minimize the number of edge-crossings and to keep the area small (presuming all coordinates are integers.)

It has been known for many years that any planar graph has a straight-line drawing without crossing in an $O(n) \times O(n)$-grid [FPP90, Sch90b]. It is also known that an $\Omega(n) \times \Omega(n)$-grid is required for some planar graphs [FPP88]. For some in subclasses of planar graphs, smaller drawings are possible. In an earlier paper, we showed that any outer-planar graph has a so-called visibility representation in an $O(\log n) \times O(n)$-grid [Bie02]. Many other papers have since dealt with drawing subclasses of planar graphs drawn in $o(n^2)$ area, such as straight-line drawings of outer-planar graphs [GR07, Fra07, DF09], and drawings of series-parallel graphs [TNU09, Fra10, Bie11].

For most of these drawing results, the output of the algorithm is a drawing that is guaranteed to have area $O(f(n))$, where $f(n)$ is some function in the number of vertices n. To show that such an algorithm is good, the usual approach has been to give an example of

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a graph that is required to have area $\Omega(g(n))$ in any drawing, for some function $g(n)$ that is close to $f(n)$. Thus, the usual approach has been to give bounds that are optimal in the worst-case, but which may be significantly too large for some graphs.

Relatively few papers exist that draw all graphs with the optimal height (or at least provably approximate it.) Specifically, a drawing of a graph is called a *drawing of height* h if all y-coordinates are in $\{1, \ldots, h\}$. We are not aware of any proof that minimizing h is NP-hard, though minimizing the area is NP-hard, at least for disconnected graphs, both for orthogonal drawings [KL82, FW91] and for straight-line drawings [KW07]. Quite closely related are *h-level drawings*, which are drawings of height h where edges must connect different levels (and for *proper* drawings, edges must connect adjacent levels.) Testing whether a graph has a proper level drawing is NP-hard [HR92], but given an h, testing whether a graph has drawing on h levels is fixed-parameter tractable in h [DFK$^+$08].

The latter paper was also among the first to prove the strong connection between the height of drawings and the so-called pathwidth $pw(G)$ of a graph G. In particular, any planar graph that has a drawing of height h has pathwidth at most h [FLW03]. However, the pathwidth is not always proportional to the minimum height: There exists a planar graph of pathwidth 3 that requires $\Omega(n)$ width and height in any planar drawing [Bie11]. But for trees, Suderman showed that the pathwidth relates closely to the optimum height: Any tree T has a planar drawing of height at most $\frac{3}{2}pw(T) - 1$ [Sud04].

In this paper, we prove a similar result to Suderman, but for a 2-connected outer-planar graph G (detailed definitions are given below.) We show that G has a flat visibility representation of height $4pw(G) - 3$, where $pw(G)$ is the pathwidth of G. Our algorithm therefore produces a height that is within a factor of 4 of the optimum. We use flat visibility representations because we find these especially easy to handle, but we also show that they can be transformed into straight-line drawings of the same height.

2 Definitions

We assume familiarity with basic graph-theoretic terms. In the following, let $G = (V, E)$ be a simple graph with n vertices V and m edges E. Throughout the paper, we assume that G is *planar*, i.e., it can be drawn without crossing. Furthermore, we assume that G is *outer-planar*, i.e., it has a drawing without crossings such that all vertices are on the *outer-face* (the infinite connected region outside the drawing.) Any finite region defined by such a drawing is called an *interior face*, and we often identify faces with the vertices and edges that are adjacent to it.

A graph is called *maximal outer-planar* if we cannot add any edges to it and retain an outer-planar simple graph. In a maximal outer-planar graph, the outer-face consists of a simple cycle of length n, and every interior face is a triangle. The *dual tree* of a maximal outer-planar graph consists of placing a vertex for every interior face and connecting two vertices if and only the corresponding faces share an edge. It is easy to see that the result is indeed a tree and has maximum degree 3.

A graph is said to have *pathwidth* k if there exists an order of the vertices $v_1, \ldots, v_n$ such that for any $j \geq k$, there are at most k vertices among $\{v_1, \ldots, v_j\}$ that have a neighbour in

$\{v_{j+1}, \ldots, v_n\}$. For trees, the pathwidth can be described using the notation of a *main path* introduced by Suderman [Sud04].

Definition 1 *Let T be a tree of pathwidth $p > 0$. A* main path *of T is a path P such that every component of $T - P$ has pathwidth at most $p - 1$.*

It is easy to see (proved by Suderman [Sud04] and even earlier by Ellis et al. [EST94]) that every tree of pathwidth $p > 0$ has a main path. Note that the main path is not unique. We often assume that a main path ends at a leaf of the dual tree, for if it doesn't, then it can simply be extended into a leaf and remains a main path.

A *drawing* of a graph consists of assigning a point or an axis-aligned box to every vertex, and a curve between the points/boxes of u and v to every edge (u, v). The drawing is called *planar* if curves of edges do not intersect curves of other edges or points/boxes of vertices other than their endpoints. We only consider planar drawings in this paper and occasionally omit "planar". The most commonly considered type of drawing is a *straight-line drawing* where vertices are represented by points and edges are drawn as straight-line segments. In this paper, we also study *visibility representations*, where vertices are represented by axis-aligned boxes and edges are drawn as horizontal or vertical straight-line segments. A visibility representation is called a *flat visibility representation* if every vertex-box is degenerated into a horizontal segment.

In all our drawings, we presume that the defining elements (i.e., points of vertices, corners of boxes of vertices, and attachment points of edges to vertex-boxes) are placed at points with integer coordinates. A drawing is said to have *width* w and *height* h if all such points are placed on the $[1, w] \times [1, h]$-grid. (Note that as opposed to some other graph drawing papers, we measure the height by the number of *rows*, i.e., horizontal lines with integer y-coordinates that are occupied by the drawing, and not by the vertical length of the minimum enclosing box. This will make some of the recursive computations simpler.)

3 Visibility representations of outer-planar graphs

We first give an overview of the algorithm to create flat visibility representations of a 2-connected outer-planar graph G.

- Convert G into a maximal outer-planar graph G' by adding edges. This can be done such that $pw(G') \leq pw(G) + 1$ [BF02].
- Let T be the dual tree of G'. It has maximum degree 3 since G' was maximal outer-planar. Moreover $pw(T) \leq pw(G') - 1$ [BF02] and therefore $pw(T) \leq pw(G)$.
- We will give a recursive algorithm to create a drawing of G' (and hence of G) whose height is at most $\max\{3, 4pw(T) - 3\} \leq 4pw(G) - 3$ as follows:
  - If T is a path, then it is very easy to create a drawing of height 2.
  - If $pw(T) \geq 1$, then we draw the graph of a main path P of T with height 2, and merge the subgraphs defined by the components of $T - P$ after drawing them recursively.

To allow the last merging step to be done with adding too much height, we will put restrictions on two vertices that form an edge on the outer-face of the subgraph as follows.

Definition 2 *Let G be a maximal outer-planar graph and let (u,v) be an edge on the outer-face, with u before v in clockwise order. Let Γ be a flat visibility representation of G. We say that $\{u,v\}$* spans the top of Γ *if the box of u occupies the top left corner, and the box of v occupies the top right corner.*

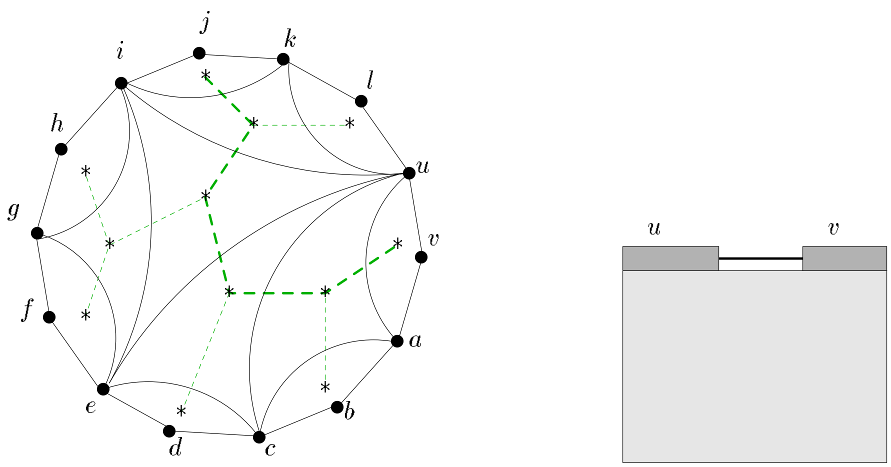

Figure 1: (Left) Example of a maximal outer-planar graph, its dual tree (dashed), and a main path (thick dashed.) (Right) A flat visibility representation where $\{u,v\}$ spans the top.

We also need the following notation. Let (u,v) be an edge on the outer-face, and let f be its adjacent interior face. We say that (u,v) *is adjacent to a main path* if there exists a main path of T that contains f. We now state the main result, which implies a recursive drawing algorithm.

Lemma 3.1 *Let G be a maximal outer-planar graph with edge (u,v) on the outer-face, with u before v in clockwise order. Let T be the dual tree of G.*

1. *There exists a flat visibility representation of G with $\{u,v\}$ spanning the top that has height* $\max\{2, 4pw(T)\}$.

2. *If (u,v) is adjacent to a main path of T, then there exists a flat visibility representation of G with $\{u,v\}$ in the top row that has height* $\max\{3, 4pw(T)-3\}$.

3. *If (u,v) is adjacent to a main path of T, then there exists a flat visibility representation of G with $\{u,v\}$ spanning the top row that has height* $\max\{4, 4pw(T)-2\}$.

Proof: As a first ingredient, we study how to draw a graph G whose dual tree T is a path $P = f_1, f_2, \ldots, f_k$. Here each f_i is a vertex of T and hence a face of G; we will use f_i for both vertex and face since the meaning should be clear from the context.

Let G_P be the graph induced by the faces $f_1, \ldots, f_k$. Create a visibility representation of G_P with height 2 in the obvious way: Draw the faces $f_1, \ldots, f_k$ as squares from left to right, and place each vertex of G_P so that it reaches the squares of all faces it belongs to. This uniquely determines the placement of all vertices except at f_1 and f_k (where the vertex of degree 2 could go on either row). We choose the placement of vertices at f_1 and f_k such that u and v end up in the same row (after possible rotation, we may assume that it is the top row.) See also Figure 2.

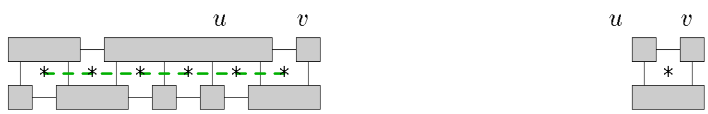

Figure 2: How to draw a graph G_P whose dual tree is a path P.

The proof of the lemma is now by induction on the pathwidth of T. In the base case $pw(T) = 0$ tree T is a singleton vertex, and the above drawing has height 2 and $\{u, v\}$ spans the top; this proves all claims. For the inductive step, we first show (2), then (3), and then (1) since each uses the other.

(2) Let $f_1, \ldots, f_k$ be a main path P to which (u, v) is adjacent; as usual we assume that it begins and ends at leaves of T. For each face f_i, $i = 1, \ldots, k$, let T_i be the subtree of $T - P$ whose root is adjacent to f_i. See also Figure 3. Note that f_1 and f_k have no such subtree since they are leaves. All other f_i's have at most one such subtree since they have two neighbours on P and degree ≤ 3. By definition of a main path, T_i has treewidth at most $pw(T) - 1$.

Draw the graph G_P formed by the faces $f_1, \ldots, f_k$ as explained above; this places $\{u, v\}$ in the top row. Let G_i be the subgraph of G for which T_i is the dual graph. Thus, G_i is a maximal outer-planar subgraph whose dual tree has pathwidth at most $pw(T) - 1$. Let (u_i, u_i) be the edge that G_i shares with face f_i, with u_i clockwise before v_i on G_i. By induction, G_i has a drawing with $\{u_i, v_i\}$ spanning the top that has height $H_i \leq \max\{2, 4pw(T_i)\} \leq \max\{2, 4pw(T) - 4\}$.

Now take the drawing of G_P and expand it vertically by adding $\max_i\{H_i\} - 1$ rows. For each $i = 2, \ldots, k$, if the drawing of G_i has W_i columns, then add W_i columns between the drawings of u_i and v_i in G_P. (Note that u_i and v_i are horizontally adjacent in G_P, since they are not incident to f_1 or f_k.) Insert the drawing of G_i into the space thus created for it, after rotating it 180° if $\{v_i, u_i\}$ is in the top row, and flipping it vertically. See Figure 3.

The drawing of G_i has height at most $\max\{2, 4pw(T) - 4\}$. When inserting G_i into G_P, we re-use the row that contains u_i and v_i, so we need to add at most $\max\{1, 4pw(T) - 5\}$ rows to the two rows of the drawing of G_P. The final height hence is $\max\{3, 4pw(T) - 3\}$ which gives the result.

(3) By (2), we know that G can be drawn with u and v in the top row, with height $\max\{3, 4pw(T) - 3\}$. We now *release u and v* by adding a row and relocating them

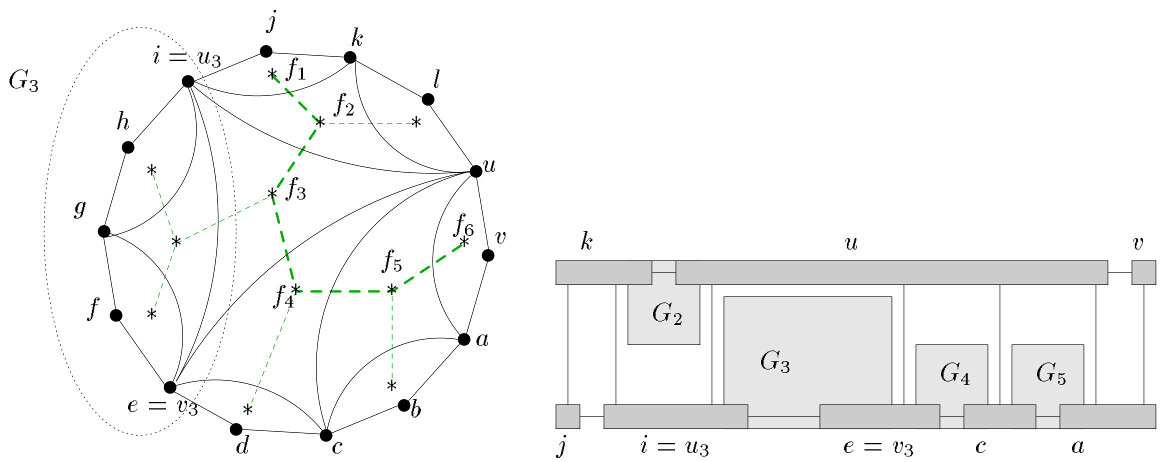

Figure 3: Definition of G_i, and how to merge it into the drawing of G_P.

into it, so that they span the new top row. (This is quite similar in spirit to the modifications introduced in our earlier paper [Bie02].)

More precisely, add a new row above the existing drawing. Move u and v into this new row, with u occupying everything from the top left corner to its rightmost column, and v occupying everything else in the top row. If x was a neighbour of u, then it either was connected to u by a vertical line (which can simply be extended to continue to the new position of u), or it was the unique vertex to the left of u in the top row. (Here we crucially use that we have a flat visibility representation, i.e., that every vertex-box has unit height.) In the latter case, x can now add a vertical line towards the new position of u, since u spans the whole range above x. Similarly we can connect any neighbour of v to the new position of v.

Thus, releasing u and v adds one unit of height and achieves that u and v span the top row. Thus the result holds by (2).

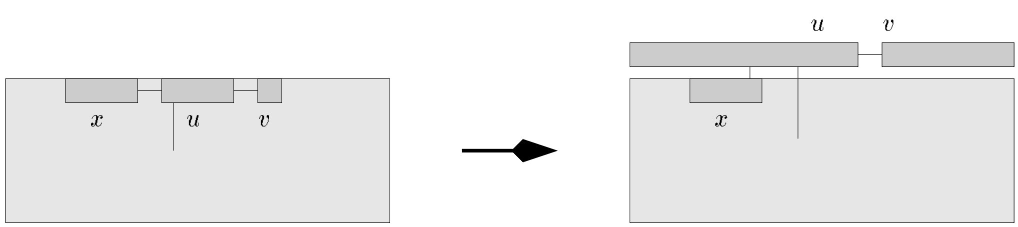

Figure 4: Releasing u and v.

(1) We will first create a drawing where u and v are in the top row, and then release them as in Case (3). To create the drawing, we proceed similarly as in case (2), but since (u, v) is not on a main path, we use a different path and therefore need more height for the subgraphs.

Let $P' = f'_1, \ldots, f'_{k'}$ be a main path of T. Let f be the interior face that is adjacent to (u, v). In tree T, there is a unique (and non-empty) shortest path that connects f to a vertex $f'_{j'}$ that belongs to the main path. We have $1 \neq j' \neq k'$, otherwise we could simply have extended the main path to f and be in case (2).

Let P be the path that consists of the path from f to $f'_{j'}$, and then continues with $f'_{j'+1}, f'_{j'+2}, \ldots, f'_{k'}$. Enumerate P as $f_1, \ldots, f_k$ with $f_1 = f$, $f_j = f'_{j'}$ and $f_k = f'_{k'}$. The drawing now proceeds exactly as in case (2), i.e., define the subtree T_i of $T - P$ that is attached to f_i, and draw its corresponding graph G_i recursively. Draw the graph G_P induced by the faces $f_1, \ldots, f_k$ in two rows, and insert the drawings of $G_2, \ldots, G_{k-1}$ after adding sufficiently many rows and columns.

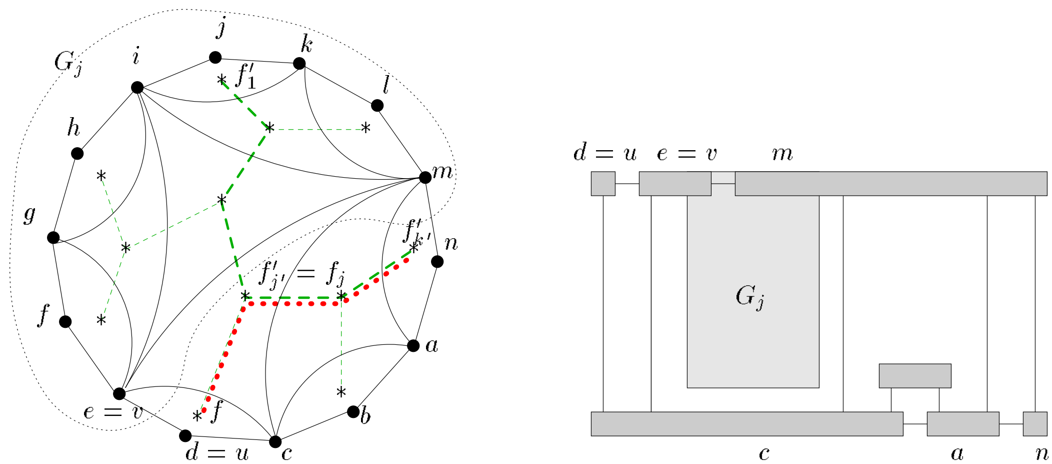

Figure 5: The example from Figure 1, but using (d, e) as edge (u, v). The path P used is marked dotted.

It remains to analyze the height. Recall that any subtree of $T - P'$ has pathwidth at most $pw(T) - 1$ since P' was a main path. For any $i \neq j$, subtree T_i is a subtree of $T - P'$ and hence has pathwidth at most $pw(T) - 1$. So for $j \neq i$, graph G_i is drawn with height at most $\max\{2, 4pw(T_i)\} \leq \max\{2, 4pw(T) - 4\}$. A special case is T_j, which contains the rest of the main path, $f'_1, \ldots, f'_{j'-1}$ and hence may well have pathwidth $pw(T)$ and then requires more height. However, the edge (u_j, v_j) (i.e., the edge shared by G_j and G_P) is incident to $f'_{j'-1}$, and hence adjacent to a main path of T_j. Therefore T_j can be drawn using case (3) with height $\max\{4, 4pw(T) - 2\}$.

Since merging these drawings into the 2-row drawing of G_P reuses one row, therefore the height of the drawing with u and v in the top row is at most $\max\{5, 4pw(T) - 1\}$. We then release u and v as in case (3) and obtained a drawing where u and v span the top with height $\max\{6, 4pw(T)\}$. This proves the result unless $pw(T) = 1$.

If $pw(T) = 1$, then T is a cater-pillar, i.e., it consists of a path with leaves attached. It is easy to draw G using three rows such that one of $\{u, v\}$ (say u) is in the top row and v (which has degree 2 since (u, v) is not adjacent to a main path) is in the middle row. Now relocate both u and v to a newly added row on the top and re-connect to their neighbours as illustrated in Figure 6; this gives a drawing of height 4 as desired.

□

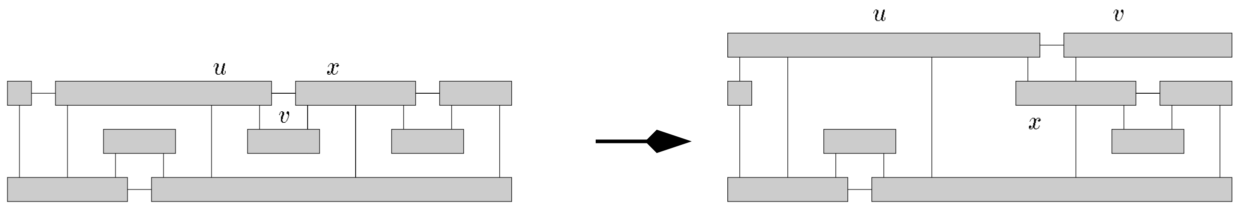

Figure 6: Drawing of a graph where the dual tree is a caterpillar, and how to transform it into one where $\{u, v\}$ span the top, even if v was not in the top row.

We summarize in a theorem:

Theorem 1 *Any 2-connected outer-planar graph G has a flat visibility representation of height $4pw(G) - 3$ and width $\frac{3}{2}(n-2)$. It can be found in linear time.*

Proof: Add edges to G until it is a maximal outer-planar G' with pathwidth $pw(G') \leq pw(G) - 1$ [BF02]. Let T be the dual tree of G'; we have $pw(T) \leq pw(G') - 1 \leq pw(G)$ [BF02]. Find a main path of T and let (u, v) be an edge on the outer-face of G' adjacent to the main path. Then draw G' with height $\max\{3, 4pw(T) - 3\}$ using case (2) of Lemma 3.1. This drawing has height at most $4pw(G) - 3$ since $pw(T) \leq pw(G)$ and $pw(G) \geq 2$ since G is 2-connected. This proves the height-bound.

For the width, it is not hard to show that it is bounded by $|T| + \ell_T - 1$, where ℓ_T is the number of leaves in T. (We consider a singleton vertex of degree 0 to be two leaves for this formula to be correct.) Namely, this holds in the drawing of a graph G_P for which the dual tree T is a path, since the width is $|T| + 1$. When merging a subgraph G_i into such a drawing, we add no new columns beyond those that already existed for G_P and G_i, but one of the leaves in T_i may not be a leaf in T, and some calculations show that the bound holds as well. Since for a maximal outer-planar graph the dual tree T has maximum degree 3, it has at most $|T|/2 + 1$ leaves. So the width is at most $\frac{3}{2}|T| = \frac{3}{2}(n-2)$ since G' has $n - 2$ interior faces.

As for the running time, computing the dual tree is straightforward. Given this tree, the pathwidth of it can be computed in linear time (see e.g. [Sch90a].) The algorithm to compute this pathwidth actually uses a rooted tree, and computes much more information about each rooted subtree, which helps since we don't need to re-compute the pathwidth of subtrees. Root the dual tree T. Compute the pathwidth of T (and all information for its rooted subtrees.) Any subtree that we need in our algorithm is actually a rooted subtree of T. Using the extra information with the pathwidth computation, it is easy to extract the pathwidth of every subtree, as well as a main path (details are left to the reader.) Hence we can look up all required information for our algorithm in constant time per subtree. All other aspects of our algorithm can clearly be implemented in linear time as well. $\square$

We have a few comments on this theorem:

- Our drawings do not preserve the planar embedding, because we "flip in" the drawings of the subgraphs G_i, and all edges of G_i except (u_i, v_i) disappear from the outer-face. If bends are allowed (i.e., if we aim for orthogonal box-drawings rather than visibility representations), then it is possible to create orthogonal box-drawings that reflect the planar embedding. The approach is similar as in [Bie02]: route the edge (u_i, v_i) "around" the drawing of G_i. The height then increases, but is still $O(pw(T))$ (details are left to the reader.)

- We made the graph maximal outerplanar because the dual tree then has maximum degree 3, which simplifies notation. The algorithm works with minor changes for any 2-connected outerplanar graph: draw a cycle in the base case, and merge multiple subgraphs into any face. This decreases the height-bound by 4 if $pw(G) \geq 2$, and may decrease the width.

4 Straight-line drawings

In our drawing algorithm, we used flat visibility representations, since the orthogonality of edges makes it easy to insert extra space for subgraphs, and the small height of boxes allows to release vertices. We now show in this section that any height-bound obtained for them also transfers to straight-line drawings, by doing a fairly straightforward transformation (which to our knowledge has not been proved before.) The height remains exactly the same (all vertices retain their y-coordinate), but the width increases much and may in fact become exponential.

Theorem 2 *Let Γ be a flat visibility representation of a graph G that has height h. Then there exists a straight-line drawing Γ' of G of height h.*

Proof: For any vertex v, use $x_l(v), x_r(v)$ and $y(v)$ to denote leftmost and rightmost x-coordinate and (unique) y-coordinate of the box that represents v in Γ. We use $X(v)$ and $Y(v)$ to denote the (initially unknown) coordinates of v in Γ'. For any vertex we enforce $Y(v) = y(v)$, which proves the height-bound.

Let $v_1, \ldots, v_n$ be the vertices sorted by $x_l(.)$, breaking ties arbitrarily. We determine $X(.)$ for each vertex by processing vertices in this order and expanding the drawing Γ'_{i-1} created for $v_1, \ldots, v_{i-1}$ into a drawing Γ'_i of $v_1, \ldots, v_i$. We maintain throughout that $Y(v) = y(v)$ for all vertices, and for any row, the left-to-right order of vertices will be the same in Γ' (as far as it has been built yet) as it was in Γ.

So presume we have determined $X(v_h)$ for all $h < i$ already. To find $X(v_i)$, we determine lower bounds for it by considering all predecessors of v_i and taking the maximum over all of them. (For each vertex v_i, the *predecessors* of v_i are the neighbours of v_i that come earlier in the order $v_1, \ldots, v_n$.) A first (trivial) lower bound for $X(v_i)$ is that it needs to be to the right of anything in row $y(v_i)$. Thus, if Γ'_{i-1} contains a vertex or part of an edge at point $(X, y(v_i))$, then we must have $X(v_i) \geq \lfloor X \rfloor + 1$.

Next consider any predecessor v_h of v_i with $y(v_h) \neq y(v_i)$. Since v_h and v_i are not in the same row, they must see each other vertically in Γ, which means that $x_r(v_h) \geq x_l(v_i)$. So if v_h has a neighbour v_k to its right in Γ, then $x_\ell(v_k) \geq x_r(v_h) \leq x_\ell(v_i)$, which implies that $k > i$, so v_k has not been added to Γ'_{i-1}. Since the order of the vertices in each row is unchanged, therefore v_h is the rightmost vertex in its row in Γ'_{i-1} and can see towards infinity on the right. But then v_h can also see the point $(+\infty, y(v_i))$, or in other words, there exists some X_h such that v_h can see all points $(X, y(v_i))$ for $X \geq X_h$. See also Figure 7. We impose the lower bound $X(v_i) \geq \lceil X_h \rceil$ on the x-coordinate of v_i.

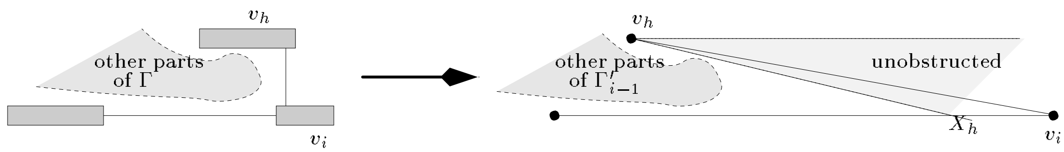

Figure 7: Transforming a flat visibility drawing into a straight-line drawing with unchanged y-coordinates.

Now let $X(v_i)$ be the smallest value that satisfies the above lower bounds (from the row $y(v_i)$ and from all predecessors of v_i in different rows.) We set $X(v_i) = 0$ if there were no such lower bounds. Directly by construction, placing v_i at $(X(v_i), y(v_i))$ allows it to be connected with straight-line segments to all its predecessors. This includes the predecessor (if any) that is in the row $y(v_i)$, since we can simply horizontally connect it to v_i. (Here is where we are using a flat visibility representation, which means that there is only one such predecessor and it is placed in the same row.) This gives a drawing Γ'_i of $v_1, \ldots, v_i$ as desired, and the result follows by induction. □

Unfortunately, while our transformation keeps the height intact, the width can increase dramatically. It is not hard to construct a flat visibility representations of height 4 and width $O(n)$ for which the resulting straight-line drawing has width $\Omega(2^n)$; see Figure 8. It remains open whether some other construction could create straight-line drawings with smaller width, perhaps by rearranging which vertex is in which row, or at the expense of some height.

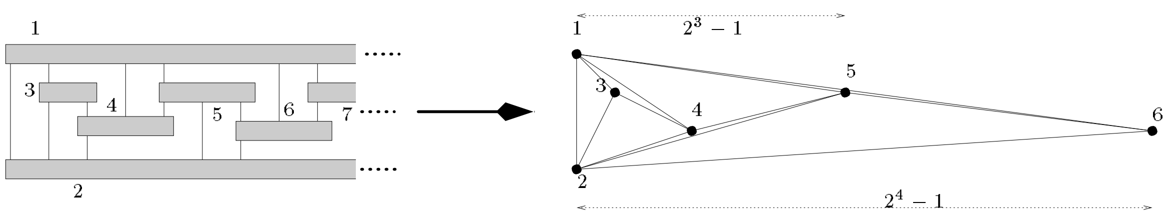

Figure 8: A flat visibility representation for which the corresponding straight-line drawing has exponential width. Vertices are numbered in the order in which they are processed. Vertex i is placed with x-coordinate $2^{i-2} - 1$ for $i \geq 3$, and leaves an edge with slope $\pm 1/(2^{i-2} - 1)$.

5 Conclusion and open problems

In this paper, we presented algorithms to draw 2-connected outer-planar graphs, with the objective of keeping the height as small as possible. While the pathwidth $pw(G)$ of such a graph G is an easy lower bound, we created drawings of height $4pw(G) - 3$; the algorithm is hence a 4-approximation algorithm for the height.

We conclude with some open problems:

- What can be said about drawing outer-planar graphs that are not 2-connected? The "obvious" approach (add edges to make the graph 2-connected, draw the result and omit the added edges) requires care that we add edges with increasing the pathwidth too much.

  The example of a caterpillar shows that some outer-planar cannot be made 2-connected without increasing the pathwidth. But how much increase is needed? In other words, if G is an outer-planar graph, can we add edges to obtain a 2-connected outer-planar graph G' such that $pw(G') \in O(pw(G))$? To our surprise, no such result appears to be known. (The closest result is by Govindran et al., which gives a 3-approximation algorithm for the pathwidth of any outerplanar graph, even not biconnected, but it is not clear whether the thus constructed path decomposition could be used to extend the graph into a 2-connected outerplanar graph while maintaining the pathwidth.)

  It is quite easy to create for any outerplanar graph G a 2-connected outerplanar supergraph G' such that $pw(G') \leq pw(T) \cdot pw(C_{\max})$, where T is the block-tree of G, and $C_{\max}$ is the 2-connected component with maximal pathwidth of G. We conjecture that in fact we can create a 2-connected outer-planar graph G' such that $pw(G') \leq O(pw(T) + pw(C_{\max}))$, but this remains open.

- Any tree has pathwidth at most $2\log_3(n) + o(\log n)$ [KS93], so our height-bound is bounded by $\approx 8\log_3(n)$. This worst-case bound is worse than the $3\log_2(n)$ bound proved earlier for outer-planar graphs [Bie02]. We have, however, not been able to find a graph where our algorithm actually uses height $\approx 8\log_3(n)$, and leave as an open problem to improve the factor of the height.

  Alternatively, can we prove lower bounds better than $pw(G)$ on the height for some graphs? Suderman showed that there are trees T, even with maximum degree 3, for which any straight-line drawing requires height at least $\lceil \frac{3}{2} pw(T) \rceil$ [Sud04]. Can we prove a similar result for outer-planar graphs? One could try an outer-planar graph for which the dual tree is Suderman's tree (but it is not clear why the lower bound for the dual tree should transfer.) Alternatively, one could try to make Suderman's tree 2-connected (but it is not clear whether this is possible without increasing the pathwidth.) So this remains open.

- Note that our height-bound depended on $pw(T)$ rather than $pw(G)$. While we know $pw(T) \leq pw(G)$, this bound is not tight: There exists a 2-connected outer-planar graph with pathwidth $2p+1$ whose dual tree has pathwidth p [CHS07]. Would it be possible to get a better bound on the height if we use the pathwidth of G itself, rather than the pathwidth of the dual tree, to guide the algorithm?

- Our bound on the width for visibility representations was $O(n)$. Can this be reduced? Obviously the width must be $O(n)$ in a visibility representation if there exists a vertex of degree $\Omega(n)$ and the height is $O(pw(G))$. But can we get a width-bound that is asymptotically less if the maximum degree Δ is smaller? For example, can we have height $O(pw(G))$, and width $O(\max\{n/pw(G), \Delta\})$?

- Are there straight-line drawings of outerplanar graphs that have height $O(pw(G))$ and polynomially bounded width? Can we bound the width by $O(n)$ or even better?
- Recall that our construction does not preserve the planar embedding. Is there a visibility representation that has height $O(pw(G))$ and preserves the planar embedding?
- What height can be achieved for series-parallel graphs? We cannot hope to create drawings of height $O(pw(G))$ for a series-parallel graph, the series-parallel graph presented by Frati [Fra10] has pathwidth $O(\log n)$, but requires $\Omega(2^{\sqrt{\log n}})$ width and height in any straight-line drawing. But can we create some drawings of series-parallel graphs where the height is a function of the pathwidth only?
- Suderman studied many different versions of leveled drawings of trees, depending on whether edges may stay within the same level, or cross multiple levels, etc. In the same spirit, one could restrict the types of visibility representations more and ask for bounds on the height. For example, what height can we achieve for flat visibility representations of outer-planar graphs if all edges must be drawn vertically? It is easy to show that this can always be achieved with $O(pw(G))$ height (see [Bie11] for various transformations among drawings), but what is the best factor that can be achieved? On the other hand, can we achieve a smaller height if we drop the "flat" requirement on the visibility representation?
- The pathwidth, while useful for graph drawing applications, is not quite the right bound for the height of a drawing. Is there another graph parameter, likely quite similar to the path width but taking some "distance to outer-face" constraints into account that captures the asymptotic height of a drawing for all planar graphs?

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