A simpler representation for $R(4,4)$

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Abstract

We look at an alternate basis for $R(4,4)$, and see how this alternate basis affects the scalar coefficients both in geometric formulas and in affine transformations in this space. In particular, the original basis used in $R(4,4)$ introduced powers of 2 into these formulas; with the new basis, most of these formulas are simpler, and do not have these powers of 2.

1 Introduction

Goldman and Mann [2] studied representations of points, lines, planes, and quadratic surfaces as well as affine transformations and perspective transformations in the Clifford algebra $R(4,4)$ for use in computer graphics. In a later paper [3], Du et al. derived closed formulas for the intersections of these objects as well as the lengths, areas, and volumes of these objects. Two bases were used in their work on $R(4,4)$, an $e_i, \bar{e}_i$ basis, which is primarily used for transformations, and a w_i, w_i^* basis, which is primarily used to represent geometric objects. Goldman and Mann followed Doran et al.'s [1] choice for the w_i,w_i^* basis, the Witt basis. While the Witt basis leads to reasonable representations for the transformations, powers of $\frac{1}{2}$ appear in the geometric constructions of Du et al. [3].

In this paper, we use an alternate relationship between the $e_i, \bar{e}_i$ basis and the w_i, w_i^* basis. Using this alternate basis for w_i, w_i^*, the powers of $\frac{1}{2}$ that appear in the geometric construction of Du et al. [3] disappear. However, as expected, this alternate basis changes the coefficients in some of the transformations in the earlier paper of Goldman and Mann [2]. In this paper, we compare the formulas that result from both bases; overall, the new basis leads to simpler formulas.

In Section 2, we review the algebra $R(4,4)$, giving the original and new bases used for the w_i, w_i^*s. In Section 3, we show how the geometric formulas of Du et al. [3] simplify when using the new basis. In Section 4, we show how the transformation formulas of Goldman and Mann [2] change when using the new basis. In both sections, we give tables comparing all the formulas of Goldman-Mann [2] and Du et al. [3]

using both basis. The derivations of the equations using the modified Witt basis are essentially the same as the derivations using the Witt basis, so we give only a few representive derivations. The goals of this paper are to show the differences in the formulas resulting from the different bases, and to present all the formulas for the alternative basis in one place.

2 The Clifford algebra $R(4,4)$

The Clifford algebra $R(4,4)$ represents 3D affine geometry using an 8-dimensional vector space. One basis for $R(4,4)$ has four basis vectors e_0, e_1, e_2, e_3 that square to $+1$, and four basis vectors $\bar{e}_0, \bar{e}_1, \bar{e}_2, \bar{e}_3$ that square to -1:

$$e_i^2 = 1,\ \bar{e}_i^2 = -1,\ (i = 0, 1, 2, 3).$$

These basis elements satisfy the following identities:

$$\begin{aligned} e_i \cdot e_j = \bar{e}_i \cdot \bar{e}_j &= 0,\ i \neq j \\ e_i \cdot \bar{e}_j &= 0, \quad \text{for all } i, j, \end{aligned}$$

and

$$\begin{aligned} e_i e_j = -e_j e_i,\ \bar{e}_i \bar{e}_j &= -\bar{e}_j \bar{e}_i,\ i \neq j \\ e_i \bar{e}_j &= -\bar{e}_j e_i, \quad \text{for all } i, j. \end{aligned}$$

To represent points and vectors in 3-dimensions and derive affine and projective transformations on R^3, Goldman-Mann used for a second basis the choice of Doran et al. [1], the Witt basis:

$$\mathsf{w}_i = \frac{e_i + \bar{e}_i}{2},\ \mathsf{w}_i^* = \frac{e_i - \bar{e}_i}{2},\ (i = 0, 1, 2, 3). \tag{1}$$

In this paper, we will explore using an alternative second basis, which we refer to as the *modified Witt basis*,

$$w_i = \frac{e_i + \bar{e}_i}{\sqrt{2}},\ w_i^* = \frac{e_i - \bar{e}_i}{\sqrt{2}},\ (i = 0, 1, 2, 3). \tag{2}$$

To distinguish between these two bases, we use $\mathsf{w}_i, \mathsf{w}_i^*$ for the Witt basis used by Goldman-Mann, and w_i, w_i^* for the modified Witt basis; at times, when both basis yield the same formula, we will state that either basis can be used and use w_i, w_i^* to represent both bases in one equation.

In the Witt basis, vectors in the 3-space are represented in the basis $\{\mathsf{w}_1, \mathsf{w}_2, \mathsf{w}_3\}$ and points have the homogeneous representation

$$\mathsf{w}_0 + x\mathsf{w}_1 + y\mathsf{w}_2 + z\mathsf{w}_3,$$

while in the modified Witt basis vectors are represented in the basis $\{w_1, w_2, w_3\}$ and points have the homogeneous representation

$$w_0 + xw_1 + yw_2 + zw_3.$$

The two bases are distinguished by the following inner products:

Witt Basis	Modified Witt Basis
$\mathsf{w}_i \cdot \mathsf{w}_j = 0$	$w_i \cdot w_j = 0$
$\mathsf{w}_i^* \cdot \mathsf{w}_j^* = 0$	$w_i^* \cdot w_j^* = 0$
$\mathsf{w}_i^* \cdot \mathsf{w}_j = \frac{1}{2}\delta_{i,j}$	$w_i^* \cdot w_j = \delta_{i,j}$

For $i \neq j$, the outer product of the both forms of the w_i, w_i^* bases share the following relationships:

$$\begin{aligned} w_i \wedge w_j = w_i w_j = -w_j w_i,\ w_i^* \wedge w_j^* = w_i^* w_j^* = -w_j^* w_i^* \\ w_i^2 = w_i \wedge w_i = 0,\ (w_i^*)^2 = w_i^* \wedge w_i^* = 0 \end{aligned}$$

We can then derive the following formulas for swapping w_i and w_i^* in both bases:

Table 1: Standard algebraic identities in Clifford algebra with Witt basis and modified Witt basis

Witt basis	Modified Witt basis
$\|\|\mathsf{u}_1 \wedge \cdots \wedge \mathsf{u}_k\|\|^2 = 2^k(\mathsf{u}_1 \wedge \cdots \wedge \mathsf{u}_k) \cdot (\mathsf{u}_k^* \wedge \cdots \wedge \mathsf{u}_1^*)$	$\|\|u_1 \wedge \cdots \wedge u_k\|\|^2 = (u_1 \wedge \cdots \wedge u_k) \cdot (u_k^* \wedge \cdots \wedge u_1^*)$
$\|\|\mathsf{u}_1^* \wedge \cdots \wedge \mathsf{u}_k^*\|\|^2 = 2^k(\mathsf{u}_1^* \wedge \cdots \wedge \mathsf{u}_k^*) \cdot (\mathsf{u}_k \wedge \cdots \wedge \mathsf{u}_1)$	$\|\|u_1^* \wedge \cdots \wedge u_k^*\|\|^2 = (u_1^* \wedge \cdots \wedge u_k^*) \cdot (u_k \wedge \cdots \wedge u_1)$
$\|\|\mathsf{v}\|\|^2 = 2(\mathsf{v} \cdot \mathsf{v}^*)$	$\|\|v\|\|^2 = (v \cdot v^*)$
$2(\mathsf{v} \cdot \mathsf{u}^*) = \|\|\mathsf{v}\|\|\|\|\mathsf{u}\|\| \cos(\theta)$	$(v \cdot u^*) = \|\|v\|\|\|\|u\|\| \cos(\theta)$
$\|\|\mathsf{n}_s^*\|\|^2 = 2\mathsf{n}_s \cdot \mathsf{n}_s^*$	$\|\|n_s^*\|\|^2 = n_s \cdot n_s^*$
$2(\mathsf{n}_1 \cdot \mathsf{n}_2^*) = \|\|\mathsf{n}_1^*\|\|\|\|\mathsf{n}_2^*\|\| cos(\theta)$	$(n_1 \cdot n_2^*) = \|\|n_1^*\|\|\|\|n_2^*\|\| \cos(\theta)$
$\mathsf{u} \times \mathsf{v} = -4(\mathsf{u}^* \wedge \mathsf{v}^*) \cdot (\mathsf{w}_1 \wedge \mathsf{w}_2 \wedge \mathsf{w}_3)$	$u \times v = -(u^* \wedge v^*) \cdot (w_1 \wedge w_2 \wedge w_3)$
$\|\|\mathsf{u} \wedge \mathsf{v}\|\|^2 = -4(\mathsf{u} \wedge \mathsf{v}) \cdot (\mathsf{v} \wedge \mathsf{u})^* = (\mathsf{u} \wedge \mathsf{v}) \cdot (\mathsf{v}^* \wedge \mathsf{u}^*) = \|\|\mathsf{u} \times \mathsf{v}\|\|^2$	$\|\|u \wedge v\|\|^2 = -(u \wedge v) \cdot (v \wedge u)^* = (u \wedge v) \cdot (v^* \wedge u^*) = \|\|u \times v\|\|^2$
$\|\|\mathsf{n}_1^* \wedge \mathsf{n}_2^*\|\|^2 = 4(\mathsf{n}_1 \wedge \mathsf{n}_2) \cdot (\mathsf{n}_2^* \wedge \mathsf{n}_1^*) = \|\|\mathsf{n}_1^* \times \mathsf{n}_2^*\|\|^2$	$\|\|n_1^* \wedge n_2^*\|\|^2 = (n_1 \wedge n_2) \cdot (n_2^* \wedge n_1^*) = \|\|n_1^* \times n_2^*\|\|^2$
$\|\|\mathsf{w}_1 \wedge \mathsf{w}_2 \wedge \mathsf{w}_3\|\| = 1$	$\|\|w_1 \wedge w_2 \wedge w_3\|\| = 1$
$\|\|\mathsf{u} \wedge \mathsf{v} \wedge \mathsf{w}\|\|^2 = \det(\mathsf{u}, \mathsf{v}, \mathsf{w})^2$	$\|\|u \wedge v \wedge w\|\|^2 = \det(u, v, w)^2$

Witt Basis	Modified Witt Basis
$\mathsf{w}_i \mathsf{w}_i^* = 1 - \mathsf{w}_i^* \mathsf{w}_i$	$w_i w_i^* = 2 - w_i^* w_i$

This swapping formula is the key formula in many derivations; we give an example of the use of these formulas for swapping w_i and w_i^* in Section 4.3, and show how the different variations result in different scalar factors in the resulting derivations.

2.1 Algebraic identities

Du et al. [3] note that in their formulation of $R(4,4)$, some standard algebraic identities in Clifford algebras "differ somewhat from those metric formulas in other Clifford algebras", and give a table of these identities. Constructing $R(4,4)$ with the modified Witt basis, these standard algebraic identities resume the more familiar form found in other Clifford algebras; a comparison of these formulas between the two variations of $R(4,4)$ appears in Table 1.

3 Geometric Constructions in $R(4,4)$

Our primary motivation for using the w_i, w_i^* basis instead of the $\mathsf{w}_i, \mathsf{w}_i^*$ basis is to simplify the coefficients in the geometric constructions of Du et al. [3]. In this section, we highlight the new formulas for a few important constructions, and then give tables comparing all the constructions in the Du et al. paper using the Witt basis to the formulas using the modified Witt basis. To begin, we look at vectors in R^3. A comparison of the two shows that the squared length formula of a vector v is simpler in modified Witt basis as compared to the Witt basis:

Witt Basis	Modified Witt Basis
$\|\mathsf{v}\|^2 = 2\mathsf{v} \cdot \mathsf{v}^*$	$\|v\|^2 = v \cdot v^*$.

Table 2: Formulas for squared length, area, and volume of the line segment, triangle, and tetrahedron associated to the corresponding blades

Object	Witt basis	Modified Witt basis
Line segment $\mathsf{l} = p_1 \wedge p_2$	$\|\|l\|\|^2 = 8(\mathsf{w}_0^* \cdot \mathsf{l}) \cdot (\mathsf{l}^* \cdot \mathsf{w}_0)$	$\|\|l\|\|^2 = (w_0^* \cdot l) \cdot (l^* \cdot w_0)$
Triangle $\pi = p_1 \wedge p_2 \wedge p_3$	$\|\|\Pi\|\|^2 = -4(\mathsf{w}_0^* \cdot \Pi) \cdot (\Pi^* \cdot \mathsf{w}_0)$	$\|\|\pi\|\|^2 = -(w_0^* \cdot \pi) \cdot (\pi^* \cdot w_0)$
Tetrahedron $\Delta = p_1 \wedge p_2 \wedge p_3 \wedge p_4$	$\|\|\Delta\|\|^2 = \frac{16}{3}(\mathsf{w}_0^* \cdot \Delta) \cdot (\Delta^* \cdot \mathsf{w}_0)$	$\|\|\Delta\|\|^2 = \frac{1}{3}(w_0^* \cdot \Delta) \cdot (\Delta^* \cdot w_0)$
Dual plane	$\|\|\Pi^*\|\| = \frac{1}{2}\|\|\mathsf{n}^*\|\|$	$\|\|\pi^*\|\| = \|\|n^*\|\|$
Dual tetrahedron	$\|\|\mathsf{l}^*\|\| = \|\|\mathsf{n}_1^* \wedge \mathsf{n}_2^*\|\|$	$\|\|l^*\|\| = \|\|n_1^* \wedge n_2^*\|\|$

3.1 Duality of $\wedge W$ and $\wedge W^*$ Subspaces

Du et al. note that $\wedge\mathsf{W}$ and $\wedge\mathsf{W}^*$ are dual spaces, and show how to map between these two spaces. In this section, we show how this mapping changes when mapping between $\wedge W$ and $\wedge W^*$.

I and I^* map between elements in W and elements in W^*. Let I and I^* be the pseudo-scalars in $\wedge\mathsf{W}$ and $\wedge\mathsf{W}^*$:

$$\mathsf{I} = \mathsf{w}_0 \wedge \mathsf{w}_1 \wedge \mathsf{w}_2 \wedge \mathsf{w}_3$$
$$\mathsf{I}^* = \mathsf{w}_0^* \wedge \mathsf{w}_1^* \wedge \mathsf{w}_2^* \wedge \mathsf{w}_3^*$$

Similarly, let I and I^* be the pseudo-scalars in $\wedge W$ and $\wedge W^*$:

$$I = w_0 \wedge w_1 \wedge w_2 \wedge w_3$$
$$I^* = w_0^* \wedge w_1^* \wedge w_2^* \wedge w_3^*$$

Because $w_i^* \cdot w_i = 1$ while $\mathsf{w}_i^* \cdot \mathsf{w}_i = \frac{1}{2}$, the mapping between the $\wedge W$ and $\wedge W^*$ spaces is simpler in the modified Witt basis than in the Witt basis:

Dual	Witt Basis	Modified Basis
dual(f)	$2^{dim(\mathsf{f})+1}\mathsf{f} \cdot \mathsf{I}^*$	$f \cdot I^*$
dual(f^*)	$2^{3-dim(\mathsf{f}^*)}\mathsf{f}^* \cdot \mathsf{I}$	$f^* \cdot I$

where f, f^* and f, f^* are geometric objects in the $\wedge\mathsf{W}$ and $\wedge\mathsf{W}^*$ spaces and $\wedge W$ and $\wedge W^*$ spaces.

3.2 Points, lines, planes, and quadrics

Du et al. used a standard homogeneous representation for points, lines and planes. For example, a homogeneous point is represented as

$$\mathsf{p} \quad = \quad p_0\mathsf{w}_0 + p_1\mathsf{w}_1 + p_2\mathsf{w}_2 + p_3\mathsf{w}_3$$

For a plane in 3-dimensions with the homogeneous implicit equation

$$S(x_0, x_1, x_2, x_3) = s_0x_0 + s_1x_1 + s_2x_2 + s_3x_3$$

where s_1, s_2, s_3, s_0 are constants, the normal to this plane is

$$\mathsf{n}_s^* = s_1\mathsf{w}_1^* + s_2\mathsf{w}_2^* + s_3\mathsf{w}_3^*,$$

and Du et al. represent planes in the dual space as

$$\Pi^* \quad = \quad s_0\mathsf{w}_0^* + s_1\mathsf{w}_1^* + s_2\mathsf{w}_2^* + s_3\mathsf{w}_3^*$$

Table 3: Equations for points, vectors in planes, and points on quadric surfaces.

Witt basis	Modified Witt basis
$\mathsf{p} \cdot \Pi_s^* = \frac{1}{2} S(x_0, x_1, x_2, x_3)$	$p \cdot \pi_s^* = S(x_0, x_1, x_2, x_3)$
$\mathsf{n}_s^* \cdot (\mathsf{q} - \mathsf{p}) = \frac{1}{2} S(1, q_1, q_2, q_3)$	$n_s^* \cdot (q - p) = S(1, q_1, q_2, q_3)$
$\mathsf{p}(p_0, p_1, p_2, p_3) \cdot \mathsf{b}_F \cdot \mathsf{p}^*(p_0, p_1, p_2, p_3) = \frac{1}{4} F(p_0, p_1, p_2, p_3)$	$p(p_0, p_1, p_2, p_3) \cdot b_F \cdot p^*(p_0, p_1, p_2, p_3) = F(p_0, p_1, p_2, p_3)$

For a quadric surface

$$F(x_0, x_1, x_2, x_3) = \sum_{i,j=0}^{3} \lambda_{i,j} x_i x_j$$

where $\lambda_{i,j} = \lambda_{j,i}$, Du et al. used the bivector representation of Parkin [4]:

$$\mathsf{b}_F = \sum_{i,j=0}^{3} \lambda_{i,j} \mathsf{w}_i^* \mathsf{w}_j.$$

For all these objects (points, vectors, lines, planes, normals, and quadric surface), their representation relative to the modified Witt basis is the same as their representation relative to the Witt basis, using w_i, w_i^* in place of $\mathsf{w}_i, \mathsf{w}_i^*$. However, there are scalar differences in the equations for testing if a point lies on one of these objects.

Table 3 gives the equations for testing whether a point p, p or a vector v, v lies in a plane Π, π. Note that the Witt basis introduces factors of $\frac{1}{2}$ when evaluating the implicit function S, while the modified Witt basis does not. While these formulas are simpler relative to the modified Witt basis, the importance of the powers of $\frac{1}{2}$ will depend on whether or not one wants the exact expression of the implicit surfaces; i.e., we are often interested in whether a point (or vector) is in the plane (the equations in Table 3 are zero) or not in the plane (the equations in Table 3 are non-zero), in which case the factor of $\frac{1}{2}$ in the equations relative to the Witt basis could be omitted.

In both representations of $R(4, 4)$, the outer product null space representation of lines and planes is formed as the outer product of two distinct points (for the representation of a line) and the outer product of three non-colinear points (for the representation of a plane). These objects can be intersected in the standard way with either representation of $R(4, 4)$. See the paper of Du et al. [3] for details on the outer product representation of lines and planes, and in particular for details on forming these objects with weighted points and points at infinity.

3.3 Intersections of lines, planes, and quadrics

There are two ways to represent lines: as the join of two points and as the intersection of two planes [5]. The intersection of two objects in $R(4, 4)$, one represented in primal form and the other in dual form, can be computed using the inner product. For a line l (l) and a plane Π (π^*), and for two planes, Π (π) and Π^* (π^*) in the Witt basis (the modified Witt basis), their intersections are given by

Witt basis	Modified Witt basis
$\mathsf{l} \cdot \Pi^*$	$l \cdot \pi^*$
$\Pi \cdot \Pi^*$	$\pi \cdot \pi^*$

In both models of $R(4, 4)$, we can also intersect lines and planes with quadric surfaces. Table 4 gives the intersection formulas in both the Witt basis and in the modified Witt basis for intersecting the z-axis with an

Table 4: Intersection of the z-axis ($l^* = w_2^* \wedge w_1^*$) and a quadric surface b_F with the Witt basis and with the modified Witt basis. $\Gamma = w_0 \wedge w_1 \wedge w_2 \wedge w_3 \wedge w_0^* \wedge w_1^* \wedge w_2^* \wedge w_3^*$ is the pseudo-scalar of $R(4,4)$

Witt basis	Modified Witt basis
$\mathsf{P}_p = (\mathsf{l}^* \wedge \mathsf{b}_F \wedge \mathsf{l}) \cdot \Gamma$	$P_p = (l^* \wedge b_F \wedge l) \cdot \Gamma$
$= (\mathsf{w}_2^* \wedge \mathsf{w}_1^* \wedge b_F \wedge \mathsf{w}_1 \wedge \mathsf{w}_2) \cdot \Gamma$	$= (w_2^* \wedge w_1^* \wedge b_F \wedge w_1 \wedge w_2) \cdot \Gamma$
$= -\frac{1}{64}(\lambda_{00}\mathsf{w}_3 \wedge \mathsf{w}_3^* - \lambda_{30}\mathsf{w}_0 \wedge \mathsf{w}_3^*$	$= -(\lambda_{00}w_3 \wedge w_3^* - \lambda_{30}w_0 \wedge w_3^*$
$-\lambda_{03}\mathsf{w}_3 \wedge \mathsf{w}_0^* + \lambda_{33}\mathsf{w}_0 \wedge \mathsf{w}_0^*)$	$-\lambda_{03}w_3 \wedge w_0^* + \lambda_{33}w_0 \wedge w_0^*)$

Table 5: Comparison of the distance between two points p_1, p_2; a point p and a line l; and a point p and a plane Π with the Witt basis and the modified Witt basis

Measure	Witt basis	Modified Witt basis
Distance between points p_1 and p_2 $dist(p_1, p_2)^2$	$2(\mathsf{p}_2 - \mathsf{p}_1) \cdot (\mathsf{p}_2^* - \mathsf{p}_1^*)$	$(p_2 - p_1) \cdot (p_2^* - p_1^*)$
Distance between point p and line l	$dist(\mathsf{p}, \mathsf{l}) = \frac{2\|\|\mathsf{p}\wedge\mathsf{l}\|\|}{\|\|\mathsf{l}\|\|}$	$dist(p, l) = \frac{\|\|p\wedge l\|\|}{\|\|l\|\|}$
Signed distance between point p and plane π^*	$sdist(\mathsf{p}, \Pi^*) = \frac{\mathsf{p}\cdot\Pi^*}{\|\|\Pi^*\|\|}$	$sdist(p, \pi^*) = \frac{p\cdot\pi^*}{\|\|\pi^*\|\|}$

arbitrary quadric surface; Du et al. extend this special case intersection to arbitrary lines by applying affine transformations, noting that affine transformations are inner and outer morphisms [3]. Note the coefficient of 1/64 when using the Witt basis compared to the coefficient of 1 in the modified Witt basis.

In both models, the intersection of a quadric surface b_F in $W^* \wedge W$ with a plane π^* in W^* is the conic curve

$$C = \pi_S \wedge B_F \wedge \pi_S^*.$$

A point p lies on a conic C if and only if $p \cdot C \cdot p^* = 0$.

3.4 Distances and Angles

Formulas for the distance between two points, or a point and a line, or a point and a plane in $R(4,4)$ with both bases are shown in Table 5. Comparisons of distances and angles between two lines and two planes in R(4, 4) with both bases are shown in Table 6. While some of these formulas are identical, others have a factor of 2 when using the Witt basis that is not present when using the modified Witt basis.

3.5 Barycentric coordinates

Barycentric coordinates represent a point as an affine combination of the vertices of a simplex. The formulas for barycentric coordinates using the modified Witt basis are the same as those represented with the Witt basis; although powers of 2 appear in the computations, these powers of two cancel in the ratios of the formula. Table 7 gives these barycentric coordinate formulas.

4 Transformations in $R(4,4)$

Goldman-Mann [2] used the rotors constructed from the generators $E_{ij},\ F_{ij},\ K_i$ of Doran et al. [1]:

$$\begin{aligned} E_{ij} &= e_i e_j \pm \bar{e}_i \bar{e}_j \\ F_{ij} &= e_i \bar{e}_j - \bar{e}_i e_j \end{aligned}$$

Table 6: Comparisons of distance and angles between two lines and two planes in R(4, 4) with Witt basis and with the new basis

Measure	Witt basis	Modified Witt basis
Distance between two skew lines l_1, l_2	$dist(\mathsf{l}_1, \mathsf{l}_2) = \frac{\mathsf{p}(\mathsf{l}_2)\cdot^*\Pi}{\lVert^*\Pi\rVert}$	$dist(l_1, l_2) = \frac{p(l_2)\cdot^*\pi}{\lVert^*\pi\rVert}$
Distance between two parallel lines l_1, l_2	$dist(\mathsf{l}_1, \mathsf{l}_2) = dist(\mathsf{p}(\mathsf{l}_2), \mathsf{l}_1)$	$dist(l_1, l_2) = dist(p(l_2), l_1)$
Angle θ between oriented lines l_1 and l_2	$\cos(\theta) = \frac{2(\mathsf{w}_0^*\cdot\mathsf{l}_1)\cdot(\mathsf{w}_0\cdot\mathsf{l}_2^*)}{\lVert\mathsf{w}_0^*\cdot\mathsf{l}_1\rVert\lVert\mathsf{w}_0\cdot\mathsf{l}_2^*\rVert}$	$\cos(\theta) = \frac{(w_0^*\cdot l_1)\cdot(w_0\cdot l_2^*)}{\lVert w_0^*\cdot l_1\rVert\lVert w_0\cdot l_2^*\rVert}$
Signed distance between parallel planes π_1, π_2	$sdist(\Pi_1, \Pi_2) = sdist(\mathsf{p}(\Pi_2), \Pi_1)$	$sdist(\pi_1, \pi_2) = sdist(p(\pi_2), \pi_1)$
Angle θ between planes π_1^* and π_2^*	$\cos(\theta) = \frac{2(\mathsf{w}_0^*\cdot(\mathsf{w}_0\wedge\Pi_1))\cdot(\mathsf{w}_0\cdot(\mathsf{w}_0^*\wedge\Pi_2^*))}{\lVert\mathsf{w}_0^*\cdot(\mathsf{w}_0\wedge\Pi_1)\rVert\lVert\mathsf{w}_0\cdot(\mathsf{w}_0^*\wedge\Pi_2^*)\rVert}$	$\cos(\theta) = \frac{(w_0^*\cdot(w_0\wedge\pi_1))\cdot(w_0\cdot(w_0^*\wedge\pi_2^*))}{\lVert w_0^*\cdot(w_0\wedge\pi_1)\rVert\lVert w_0\cdot(w_0^*\wedge\pi_2^*)\rVert}$

Table 7: Table of barycentric coordinate formulas; these formulas for the same for both the Witt basis and the modified Witt basis.

Barycentric coordinates	Formula
$p = b_1p_1 + b_2p_2,\ b_1 + b_2 = 1$	$b_i = \frac{(-1)^{i+1}(p\wedge p_j)\cdot(p_1\wedge p_2)^*}{(p_1\wedge p_2)\cdot(p_1\wedge p_2)^*}$
$p = b_1p_1 + b_2p_2 + b_3p_3,\ b_1 + b_2 + b_3 = 1$	$b_i = \frac{(p\wedge p_j\wedge p_k)\cdot(p_1\wedge p_2\wedge p_3)^*}{(p_1\wedge p_2\wedge p_3)\cdot(p_1\wedge p_2\wedge p_3)^*}$
$p = b_1p_1 + b_2p_2 + b_3p_3 + b_4p_4,$ $b_1 + b_2 + b_3 + b_4 = 1$	$b_i = \frac{(p\wedge p_j\wedge p_k\wedge p_l)\cdot(p_1\wedge p_2\wedge p_3\wedge p_4)^*}{(p_1\wedge p_2\wedge p_3\wedge p_4)\cdot(p_1\wedge p_2\wedge p_3\wedge p_4)^*}$

$$K_i \quad = \quad \tfrac{1}{2}F_{ii} = e_i\bar{e}_i.$$

When applied to points (which are represented in the $\mathsf{w}_i, \mathsf{w}_i^*$ basis of Equation 1), Goldman-Mann showed that rotor generated by K_i is non-uniform scaling in the w_i-direction, the rotor generated by E_{ij} is rotation in the $\mathsf{w}_i\mathsf{w}_j$-plane, and the rotor generated by F_{ij} is a scissors shear in the $\mathsf{w}_i, \mathsf{w}_j$-plane.

When applying these rotors to points represented in the w_i, w_i^* basis of Equation 2, we obtain the same equations as when applying these rotors to points represented in the $\mathsf{w}_i, \mathsf{w}_i^*$ basis; i.e., there were no powers of two distinctions between these formulas. We illustrate this result for K_i; proofs for E_{ij} and F_{ij} are similar.

4.1 $K_i = e_i\bar{e}_i$: non-uniform scaling in the w_i-direction

Following the derivation of Goldman-Mann, we have $e^{\theta e_i\bar{e}_i} = \cosh(\theta) + \sinh(\theta)e_i\bar{e}_i$, since $(e_i\bar{e}_i)^2 = 1$. Applying the rotor $e^{\theta e_i\bar{e}_i}$ to e_i and $\bar{e}_i$,

$$\begin{aligned}&(\cosh(\theta) - \sinh(\theta)e_i\bar{e}_i)e_i(\cosh(\theta) + \sinh(\theta)e_i\bar{e}_i)\\ &= (\cosh^2(\theta) + \sinh^2(\theta))e_i + 2\sinh(\theta)\cosh(\theta)\bar{e}_i\end{aligned} \tag{3}$$

and

$$\begin{aligned}&(\cosh(\theta) - \sinh(\theta)e_i\bar{e}_i)\bar{e}_i(\cosh(\theta) + \sinh(\theta)e_i\bar{e}_i)\\ &= (\cosh^2(\theta) + \sinh^2(\theta))\bar{e}_i + 2\sinh(\theta)\cosh(\theta)e_i\end{aligned} \tag{4}$$

Adding equations 3 and 4 and dividing by 2 yields

$$\begin{aligned}&(\cosh(\theta) - \sinh(\theta)e_i\bar{e}_i)\mathsf{w}_i(\cosh(\theta) + \sinh(\theta)e_i\bar{e}_i)\\ &\quad = (\cosh^2(\theta) + \sinh^2(\theta) + 2\sinh(\theta)\cosh(\theta))\mathsf{w}_i\\ &\quad = (\cosh(2\theta) + \sinh(2\theta))\mathsf{w}_i = e^{2\theta}\mathsf{w}_i\end{aligned}$$

Table 8: Other nonsingular linear transformations on $R(4,4)$ with old basis and new basis

	Witt	Modified Witt
Reflection	$2\mathsf{w}_i^* \wedge \mathsf{w}_i$	$w_i^* \wedge w_i$
Classical shear	$\mathsf{w}_i^* \mathsf{w}_j$, $i \neq j$	$\frac{1}{2} w_i^* w_j$, $i \neq j$

while adding equations 3 and 4 and dividing by $\sqrt{2}$ yields

$$\begin{aligned}
&(\cosh(\theta) - \sinh(\theta) e_i \bar{e}_i) w_i (\cosh(\theta) + \sinh(\theta) e_i \bar{e}_i) \\
&\quad = (\cosh^2(\theta) + \sinh^2(\theta) + 2\sinh(\theta)\cosh(\theta)) w_i \\
&\quad = (\cosh(2\theta) + \sinh(2\theta)) w_i = e^{2\theta} w_i
\end{aligned}$$

For $i \neq j$, e_j, $\bar{e}_j$ commutes with $e_i \bar{e}_i$, so in either basis

$$(\cosh(\theta) - \sinh(\theta) e_i \bar{e}_i) w_j (\cosh(\theta) + \sinh(\theta) e_i \bar{e}_i) = w_j$$

Subtracting equations 3 and 4 and dividing by $\sqrt{2}$ gives the applications of the rotor $e^{\theta e_i \bar{e}_i}$ to elements in the space W^*:

$$\begin{aligned}
&(\cosh(\theta) - \sinh(\theta) e_i \bar{e}_i) w_i^* (\cosh(\theta) + \sinh(\theta) e_i \bar{e}_i) \\
&\quad = (\cosh(2\theta) - \sinh(2\theta)) w_i^* \\
&\quad = (\cosh(-2\theta) - \sinh(-2\theta)) w_i^* = e^{-2\theta} w_i^*.
\end{aligned}$$

For $i \neq j$, e_j, $\bar{e}_j$ commute with $e_i \bar{e}_i$, so in either basis

$$(\cosh(\theta) - \sinh(\theta) e_i \bar{e}_i) w_j^* (\cosh(\theta) + \sinh(\theta) e_i \bar{e}_i) = w_j^*$$

Similar derivations show that the rotors generated by E_{ij} and F_{ij} operate in the same manner on both the $\mathsf{w}_i, \mathsf{w}_i^*$ basis and the w_i, w_i^* basis, since in all three cases, the rotors have the same effect on the $e_i, \bar{e}_i$ basis, and our choice of w or w just results in combining the results on e_i and $\bar{e}_i$ with a factor of $\frac{1}{2}$ or $\frac{1}{\sqrt{2}}$.

4.2 Other Nonsingular Linear Transformations in $R(4,4)$: Reflection and Classical Shear

Goldman-Mann [2] derived two nonsingular linear transformations for reflection and classical shear using the Witt basis; we rederive these formulas for the modified Witt basis. A comparison of these two sets of formulas is shown in Table 8. As examples of how the powers of 2 differ for the transformation formulas using the Witt basis and the modified Witt basis, we give a detailed derivation of reflection with the Witt basis and with the modified Witt basis.

4.3 Reflection: $2\mathsf{w}_i^* \wedge \mathsf{w}_i$ *vs* $w_i^* \wedge w_i$

We reproduce here a variation of the proof that the rotor $R_i = e_i \bar{e}_i$ is a reflection [3] to show an example of the use of the formula for reversing $w_i^* w_i$. In the Witt basis we have

$$\begin{aligned}
e_i \bar{e}_i &= (\mathsf{w}_i + \mathsf{w}_i^*)(\mathsf{w}_i - \mathsf{w}_i^*) \\
&= \mathsf{w}_i^* \mathsf{w}_i - \mathsf{w}_i \mathsf{w}_i^* = 2\mathsf{w}_i^* \wedge \mathsf{w}_i
\end{aligned}$$

To see that R_i is a reflection, we first show that

$$\begin{aligned}
e_i \mathsf{w}_i e_i = \mathsf{w}_i^*, &\quad e_i \mathsf{w}_i^* e_i = \mathsf{w}_i \\
\bar{e}_i \mathsf{w}_i \bar{e}_i = \mathsf{w}_i^*, &\quad \bar{e}_i \mathsf{w}_i^* \bar{e}_i = \mathsf{w}_i.
\end{aligned}$$

Expanding the first of these equations,

$$\begin{aligned} e_i \mathsf{w}_i e_i &= (\mathsf{w}_i + \mathsf{w}_i^*)\mathsf{w}_i(\mathsf{w}_i + \mathsf{w}_i^*) = \mathsf{w}_i^*\mathsf{w}_i(\mathsf{w}_i + \mathsf{w}_i^*) \\ &= \mathsf{w}_i^*\mathsf{w}_i\mathsf{w}_i^* = (1 - \mathsf{w}_i\mathsf{w}_i^*)\mathsf{w}_i^* = \mathsf{w}_i^*. \end{aligned} \tag{5}$$

The proofs for the other three equations are similar.

We now see that $R_i = e_i\bar{e}_i$ is a reflection:

$$\begin{aligned} R_i^{-1}\mathsf{w}_i R_i = (-\bar{e}_i e_i)\mathsf{w}_i(e_i\bar{e}_i) = -\mathsf{w}_i \\ R_i^{-1}\mathsf{w}_i^* R_i = (-\bar{e}_i e_i)\mathsf{w}_i^*(e_i\bar{e}_i) = -\mathsf{w}_i^* \end{aligned}$$

Writing the rotor $e_i\bar{e}_i$ in the modified Witt basis, we have

$$\begin{aligned} e_i\bar{e}_i &= \frac{\sqrt{2}(w_i + w_i^*)}{2}\frac{\sqrt{2}(w_i - w_i^*)}{2} \\ &= \frac{1}{2}(w_i^* w_i - w_i w_i^*) = w_i^* \wedge w_i \end{aligned}$$

Thus, expanding $e_i, \bar{e}_i$ in the two variants of the Witt basis introduces different powers of 2.

The proof showing that $e_i\bar{e}_i$ is a reflection in the modified Witt basis is essentially the same as the proof in the Witt basis. But the expansion of $e_i w_i e_i$ is

$$\begin{aligned} e_i w_i e_i &= \frac{\sqrt{2}(w_i + w_i^*)}{2} w_i \frac{\sqrt{2}(w_i + w_i^*)}{2} = \frac{1}{2} w_i^* w_i (w_i + w_i^*) \\ &= \frac{1}{2} w_i^* w_i w_i^* = \frac{1}{2}(2 - w_i w_i^*) w_i^* = w_i^*. \end{aligned} \tag{6}$$

Comparing Equation 5 to Equation 6 gives an example of the two variations of the formula for swapping w_i, w_i^*; in the example in these two equations, the powers of two cancel in the modified Witt basis, yielding the same formula for reflection when the rotor R_i is written as the geometric product of $e_i, \bar{e}_i$, although when expressing R_i as the outer product of w_i^*, w_i, a factor of 2 appears when using the Witt basis.

4.4 Transformations relative to w, w^*

Goldman-Mann [2] rewrite rotors to get an expression in terms of the basis $\mathsf{w}_i, \mathsf{w}_j, \mathsf{w}_i^*, \mathsf{w}_j^*$. The derivations using the modified Witt basis are essentially the same the derivations using the Witt basis in Goldman and Mann's paper; Tables 9 and 10 give a comparison of transformations using the Witt basis $\mathsf{w}_i, \mathsf{w}_j, \mathsf{w}_i^*, \mathsf{w}_j^*$ and the modified Witt basis w_i, w_j, w_i^*, w_j^*. With the exception of shear, the formulas in the modified Witt basis are simpler (do not have factors of 2) than those in the Witt basis.

5 Conclusions

In this paper, gave a variation on an earlier $R(4,4)$ model [2, 3]. The only difference between our variation and the earlier model of $R(4,4)$ is in a scaling factor for one of the bases. With this new basis, the coefficients in the geometric formulas are mostly unity (as compared to powers of 2 when using the original basis). In addition, the coefficients in the transformation formulas are also simpler.

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Table 9: Blades and the corresponding rotors and transformations on R^3 based on classical shear in R^4 with old and new bases

Blade	w-Rotor	w-Rotor	Transformation
$w_i^* \wedge w_j$ $i \neq j$ $i, j \neq 0$	$1 + t\mathsf{w}_i^* \wedge \mathsf{w}_j$	$1 + \frac{1}{2}tw_i^* \wedge w_j$	Classical shear in the w_iw_j-plane (Only w_i-direction changes)
$w_0^* \wedge w_j$ $j \neq 0$	$1 + t\mathsf{w}_0^* \wedge \mathsf{w}_j$	$1 + \frac{1}{2}tw_0^* \wedge w_j$	Classical shear in the w_iw_j-plane (Only w_i-direction changes)
$w_j^* \wedge w_0$ $j \neq 0$	$1 - \mathsf{w}_j^* \wedge \mathsf{w}_0$	$1 - \frac{1}{2}w_j^* \wedge w_0$	Pseudo-perspective normal to w_j-direction
$w_i^* \wedge w_i$ $i \neq 0$	$c + 2s\mathsf{w}_i^* \wedge \mathsf{w}_i$	$c + sw_i^* \wedge w_i$	Nonuniform scaling by $e^{2\theta}$ in the w_i-direction
$w_0^* \wedge w_0$	$c + 2s\mathsf{w}_0^* \wedge \mathsf{w}_0$	$c + sw_0^* \wedge w_0$	Uniform scaling of points by $e^{-2\theta}$

Table 10: Rotors for rotation and scissors shear ($s = \sin(\theta),\ c = \cos(\theta)$); Scissor Shear Generator: $e_i\bar{e}_j - \bar{e}_ie_j$. Rotation Generator: $e_ie_j - \bar{e}_i\bar{e}_j$.

Transformation	w-Rotor	w-Rotor
Rotation by 2θ in w_iw_j-plane	$c^2 + 2sc(\mathsf{w}_i^* \wedge \mathsf{w}_j - \mathsf{w}_j^* \wedge \mathsf{w}_i)$ $+4s^2(\mathsf{w}_i^* \wedge \mathsf{w}_i)(\mathsf{w}_j^* \wedge \mathsf{w}_i)$	$c^2 + sc(w_i^* \wedge w_j - w_j^* \wedge w_i)$ $+s^2(w_i^* \wedge w_i)(w_j^* \wedge w_i)$
Scissors shear in w_iw_j-plane	$c^2 + 2sc(\mathsf{w}_i^* \wedge \mathsf{w}_j + \mathsf{w}_j^* \wedge \mathsf{w}_i)$ $-4s^2(\mathsf{w}_i^* \wedge \mathsf{w}_i)(\mathsf{w}_j^* \wedge \mathsf{w}_i)$	$c^2 + sc(w_i^* \wedge w_j + w_j^* \wedge w_i)$ $-s^2(w_i^* \wedge w_i)(w_j^* \wedge w_i)$

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