A simpler representation for $R(4, 4)$

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Abstract

We look at an alternate basis for $R(4, 4)$, and see how this alternate basis affects the scalar coefficients both in geometric formulas and in affine transformations in this space. In particular, the original basis used in $R(4, 4)$ introduced powers of 2 into these formulas; with the new basis, most of these formulas are simpler, and do not have these powers of 2.

1 Introduction

Goldman and Mann [2] studied representations of points, lines, planes, and quadratic surfaces as well as affine transformations and perspective transformations in the Clifford algebra $R(4, 4)$ for use in computer graphics. In a later paper [3], Du et al. derived closed formulas for the intersections of these objects as well as the lengths, areas, and volumes of these objects. Two bases were used in their work on $R(4, 4)$, an $e_i, \bar{e}_i$ basis, which is primarily used for transformations, and a $w_i, w^*_i$ basis, which is primarily used to represent geometric objects. Goldman and Mann followed Doran et al.’s [1] choice for the $w_i, w^*_i$ basis, the Witt basis. While the Witt basis leads to reasonable representations for the transformations, powers of $\frac{1}{2}$ appear in the geometric constructions of Du et al. [3].

In this paper, we use an alternate relationship between the $e_i, \bar{e}_i$ basis and the $w_i, w^*_i$ basis. Using this alternate basis for $w_i, w^*_i$, the powers of $\frac{1}{2}$ that appear in the geometric construction of Du et al. [3] disappear. However, as expected, this alternate basis changes the coefficients in some of the transformations in the earlier paper of Goldman and Mann [2]. In this paper, we compare the formulas that result from both bases; overall, the new basis leads to simpler formulas.

In Section 2, we review the algebra $R(4, 4)$, giving the original and new bases used for the $w_i, w^*_i$s. In Section 3, we show how the geometric formulas of Du et al. [3] simplify when using the new basis. In Section 4, we show how the transformation formulas of Goldman and Mann [2] change when using the new basis. In both sections, we give tables comparing all the formulas of Goldman-Mann [2] and Du et al. [3]
using both basis. The derivations of the equations using the modified Witt basis are essentially the same as the derivations using the Witt basis, so we give only a few representative derivations. The goals of this paper are to show the differences in the formulas resulting from the different bases, and to present all the formulas for the alternative basis in one place.

2 The Clifford algebra $R(4, 4)$

The Clifford algebra $R(4, 4)$ represents 3D affine geometry using an 8-dimensional vector space. One basis for $R(4, 4)$ has four basis vectors $e_0, e_1, e_2, e_3$ that square to $+1$, and four basis vectors $\bar{e}_0, \bar{e}_1, \bar{e}_2, \bar{e}_3$ that square to $-1$:

$$e_i^2 = 1, \quad \bar{e}_i^2 = -1, \quad (i = 0, 1, 2, 3).$$

These basis elements satisfy the following identities:

$$e_i \cdot e_j = \bar{e}_i \cdot \bar{e}_j = 0, \quad i \neq j$$
$$e_i \cdot \bar{e}_j = 0, \quad \text{for all } i, j,$$

and

$$e_i e_j = -e_j e_i, \quad \bar{e}_i \bar{e}_j = -\bar{e}_j \bar{e}_i, \quad i \neq j$$
$$e_i \bar{e}_j = -\bar{e}_j e_i, \quad \text{for all } i, j.$$

To represent points and vectors in 3-dimensions and derive affine and projective transformations on $R^3$, Goldman-Mann used for a second basis the choice of Doran et al. [1], the Witt basis:

$$w_i = \frac{e_i + \bar{e}_i}{2}, \quad w_i^* = \frac{e_i - \bar{e}_i}{2}, \quad (i = 0, 1, 2, 3). \quad (1)$$

In this paper, we will explore using an alternative second basis, which we refer to as the modified Witt basis,

$$w_i = \frac{e_i + \bar{e}_i}{\sqrt{2}}, \quad w_i^* = \frac{e_i - \bar{e}_i}{\sqrt{2}}, \quad (i = 0, 1, 2, 3). \quad (2)$$

To distinguish between these two bases, we use $w_i, w_i^*$ for the Witt basis used by Goldman-Mann, and $w_i, w_i^*$ for the modified Witt basis; at times, when both basis yield the same formula, we will state that either basis can be used and use $w_i, w_i^*$ to represent both bases in one equation.

In the Witt basis, vectors in the 3-space are represented in the basis $\{w_1, w_2, w_3\}$ and points have the homogeneous representation

$$w_0 + xw_1 + yw_2 + zw_3,$$

while in the modified Witt basis vectors are represented in the basis $\{w_1, w_2, w_3\}$ and points have the homogeneous representation

$$w_0 + xw_1 + yw_2 + zw_3.$$

The two bases are distinguished by the following inner products:

\[
\begin{array}{cc}
\text{Witt Basis} & \text{Modified Witt Basis} \\
\hline
w_i \cdot w_j = 0 & w_i^* \cdot w_j = 0 \\
w_i^* \cdot w_j^* = 0 & w_i^* \cdot w_j^* = 0 \\
w_i^* \cdot w_j = \frac{1}{2} \delta_{i,j} & w_i^* \cdot w_j = \delta_{i,j}
\end{array}
\]

For $i \neq j$, the outer product of the both forms of the $w_i, w_i^*$ bases share the following relationships:

$$w_i \wedge w_j = w_i w_j = -w_j w_i, \quad w_i^* \wedge w_j^* = w_j^* w_i^* = -w_i^* w_j^*$$
$$w_i^2 = w_i \wedge w_i = 0, \quad (w_i^*)^2 = w_i^* \wedge w_i^* = 0$$

We can then derive the following formulas for swapping $w_i$ and $w_i^*$ in both bases:
This swapping formula is the key formula in many derivations; we give an example of the use of these formulas in Clifford algebras: a comparison of these formulas between the two variations of Du et al. [3] note that in their formulation of (4,4), some standard algebraic identities in Clifford algebras “differ somewhat from those metric formulas in other Clifford algebras”, and give a table of these identities.

2.1 Algebraic identities

Du et al. [3] note that in their formulation of (4,4), some standard algebraic identities in Clifford algebras “differ somewhat from those metric formulas in other Clifford algebras”, and give a table of these identities. Constructing (4,4) with the modified Witt basis, these standard algebraic identities resume the more familiar form found in other Clifford algebras; a comparison of these formulas between the two variations of (4,4) appears in Table 1.

3 Geometric Constructions in (4,4)

Our primary motivation for using the $w_i, w^*_i$ basis instead of the $w_i, w^*_i$ basis is to simplify the coefficients in the geometric constructions of Du et al. [3]. In this section, we highlight the new formulas for a few important constructions, and then give tables comparing all the constructions in the Du et al. paper using the Witt basis to the formulas using the modified Witt basis. To begin, we look at vectors in $R^3$. A comparison of the two shows that the squared length formula of a vector $v$ is simpler in modified Witt basis as compared to the Witt basis:

$$||v||^2 = 2(v \cdot v^*)$$

<table>
<thead>
<tr>
<th>Witt basis</th>
<th>Modified Witt basis</th>
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</thead>
<tbody>
<tr>
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<td>$</td>
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<tr>
<td>$2(v \cdot v^*) =</td>
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<tr>
<td>$</td>
<td></td>
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<tr>
<td>$2(n_1 \cdot n^*_2) =</td>
<td></td>
</tr>
<tr>
<td>$u \times v = -4(u^* \wedge v^*) \cdot (w_1 \wedge w_2 \wedge w_3)$</td>
<td>$u \times v = -(u^* \wedge v^*) \cdot (w_1 \wedge w_2 \wedge w_3)$</td>
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</table>

This swapping formula is the key formula in many derivations; we give an example of the use of these formulas for swapping $w_i$ and $w^*_i$ in Section 4.3, and show how the different variations result in different scalar factors in the resulting derivations.

<table>
<thead>
<tr>
<th>Witt Basis</th>
<th>Modified Witt Basis</th>
</tr>
</thead>
<tbody>
<tr>
<td>$w_iw^<em>_i = 1 - w_iw^</em>_i$</td>
<td>$w_iw^<em>_i = 2 - w_iw^</em>_i$</td>
</tr>
</tbody>
</table>

Table 1: Standard algebraic identities in Clifford algebra with Witt basis and modified Witt basis.
Table 2: Formulas for squared length, area, and volume of the line segment, triangle, and tetrahedron associated to the corresponding blades

<table>
<thead>
<tr>
<th>Object</th>
<th>Witt basis</th>
<th>Modified Witt basis</th>
</tr>
</thead>
<tbody>
<tr>
<td>Line segment $l = p_1 \wedge p_2$</td>
<td>$</td>
<td></td>
</tr>
<tr>
<td>Triangle $\pi = p_1 \wedge p_2 \wedge p_3$</td>
<td>$</td>
<td></td>
</tr>
<tr>
<td>Tetrahedron $\Delta = p_1 \wedge p_2 \wedge p_3 \wedge p_4$</td>
<td>$</td>
<td></td>
</tr>
<tr>
<td>Dual plane</td>
<td>$</td>
<td></td>
</tr>
<tr>
<td>Dual tetrahedron</td>
<td>$</td>
<td></td>
</tr>
</tbody>
</table>

### 3.1 Duality of $\wedge W$ and $\wedge W^*$ Subspaces

Du et al. note that $\wedge W$ and $\wedge W^*$ are dual spaces, and show how to map between these two spaces. In this section, we show how this mapping changes when mapping between $\wedge W$ and $\wedge W^*$.

$l$ and $l^*$ map between elements in $W$ and elements in $W^*$. Let $l$ and $l^*$ be the pseudo-scalars in $\wedge W$ and $\wedge W^*$:

\[
l = w_0 \wedge w_1 \wedge w_2 \wedge w_3 \\
l^* = w_0^* \wedge w_1^* \wedge w_2^* \wedge w_3^*
\]

Similarly, let $I$ and $I^*$ be the pseudo-scalars in $\wedge W$ and $\wedge W^*$:

\[
I = w_0 \wedge w_1 \wedge w_2 \wedge w_3 \\
I^* = w_0^* \wedge w_1^* \wedge w_2^* \wedge w_3^*
\]

Because $w_i^* \cdot w_i = 1$ while $w_i^* \cdot w_i = \frac{1}{2}$, the mapping between the $\wedge W$ and $\wedge W^*$ spaces is simpler in the modified Witt basis than in the Witt basis:

<table>
<thead>
<tr>
<th>Dual</th>
<th>Witt Basis</th>
<th>Modified Basis</th>
</tr>
</thead>
<tbody>
<tr>
<td>dual($f$)</td>
<td>$2^{dim(f)} + f \cdot l^*$</td>
<td>$f \cdot I^*$</td>
</tr>
<tr>
<td>dual($f^*$)</td>
<td>$2^{3-dim(f)} f^* \cdot I$</td>
<td>$f^* \cdot I$</td>
</tr>
</tbody>
</table>

where $f, f^*$ and $f, f^*$ are geometric objects in the $\wedge W$ and $\wedge W^*$ spaces and $\wedge W$ and $\wedge W^*$ spaces.

### 3.2 Points, lines, planes, and quadrics

Du et al. used a standard homogeneous representation for points, lines and planes. For example, a homogeneous point is represented as

\[
\mathbf{p} = p_0 w_0 + p_1 w_1 + p_2 w_2 + p_3 w_3
\]

For a plane in 3-dimensions with the homogeneous implicit equation

\[
S(x_0, x_1, x_2, x_3) = s_0 x_0 + s_1 x_1 + s_2 x_2 + s_3 x_3
\]

where $s_1, s_2, s_3, s_0$ are constants, the normal to this plane is

\[
n_1^* = s_1 w_1^* + s_2 w_2^* + s_3 w_3^* ,
\]

and Du et al. represent planes in the dual space as

\[
\Pi^* = s_0 w_0^* + s_1 w_1^* + s_2 w_2^* + s_3 w_3^* 
\]
Table 3: Equations for points, vectors in planes, and points on quadric surfaces.

<table>
<thead>
<tr>
<th></th>
<th>Witt basis</th>
<th>Modified Witt basis</th>
</tr>
</thead>
<tbody>
<tr>
<td>( p \cdot \Pi^* )</td>
<td>( \frac{1}{2} S(x_0, x_1, x_2, x_3) )</td>
<td>( p \cdot \pi^* = S(x_0, x_1, x_2, x_3) )</td>
</tr>
<tr>
<td>( n^*_i \cdot (q - p) )</td>
<td>( \frac{1}{2} S(1, q_1, q_2, q_3) )</td>
<td>( n^*_i \cdot (q - p) = S(1, q_1, q_2, q_3) )</td>
</tr>
<tr>
<td>( p(p_0, p_1, p_2, p_3) \cdot b_F \cdot p^*(p_0, p_1, p_2, p_3) )</td>
<td>( \frac{1}{4} F(p_0, p_1, p_2, p_3) )</td>
<td>( p(p_0, p_1, p_2, p_3) \cdot b_F \cdot p^*(p_0, p_1, p_2, p_3) = F(p_0, p_1, p_2, p_3) )</td>
</tr>
</tbody>
</table>

For a quadric surface

\[
F(x_0, x_1, x_2, x_3) = \sum_{i,j=0}^{3} \lambda_{i,j} x_i x_j
\]

where \( \lambda_{i,j} = \lambda_{j,i} \), Du et al. used the bivector representation of Parkin [4]:

\[
b_F = \sum_{i,j=0}^{3} \lambda_{i,j} w_i^* w_j.
\]

For all these objects (points, vectors, lines, planes, normals, and quadric surface), their representation relative to the modified Witt basis is the same as their representation relative to the Witt basis, using \( w_i, w_i^* \) in place of \( w_i, w_i^* \). However, there are scalar differences in the equations for testing if a point lies on one of these objects.

Table 3 gives the equations for testing whether a point \( p, p \) or a vector \( v, v \) lies in a plane \( \Pi, \pi \). Note that the Witt basis introduces factors of \( \frac{1}{2} \) when evaluating the implicit function \( S \), while the modified Witt basis does not. While these formulas are simpler relative to the modified Witt basis, the importance of the powers of \( \frac{1}{2} \) will depend on whether or not one wants the exact expression of the implicit surfaces; i.e., we are often interested in whether a point (or vector) is in the plane (the equations in Table 3 are zero) or not in the plane (the equations in Table 3 are non-zero), in which case the factor of \( \frac{1}{2} \) in the equations relative to the Witt basis could be omitted.

In both representations of \( R(4, 4) \), the outer product null space representation of lines and planes is formed as the outer product of two distinct points (for the representation of a line) and the outer product of three non-collinear points (for the representation of a plane). These objects can be intersected in the standard way with either representation of \( R(4, 4) \). See the paper of Du et al. [3] for details on the outer product representation of lines and planes, and in particular for details on forming these objects with weighted points and points at infinity.

3.3 Intersections of lines, planes, and quadrics

There are two ways to represent lines: as the join of two points and as the intersection of two planes [5]. The intersection of two objects in \( R(4, 4) \), one represented in primal form and the other in dual form, can be computed using the inner product. For a line \( l \) and a plane \( \Pi, \pi \), and for two planes, \( \Pi, \pi \) and \( \Pi^*, \pi^* \) in the Witt basis (the modified Witt basis), their intersections are given by

<table>
<thead>
<tr>
<th></th>
<th>Witt basis</th>
<th>Modified Witt basis</th>
</tr>
</thead>
<tbody>
<tr>
<td>( 1 \cdot \Pi^* )</td>
<td>( l \cdot \pi^* )</td>
<td>( l \cdot \pi^* )</td>
</tr>
<tr>
<td>( \Pi \cdot \Pi^* )</td>
<td>( \pi \cdot \pi^* )</td>
<td>( \pi \cdot \pi^* )</td>
</tr>
</tbody>
</table>

In both models of \( R(4, 4) \), we can also intersect lines and planes with quadric surfaces. Table 4 gives the intersection formulas in both the Witt basis and in the modified Witt basis for intersecting the \( z \)-axis with an
3.4 Distances and Angles

Formulas for the distance between two points, or a point and a line, or a point and a plane in $R(4, 4)$ with both bases are shown in Table 5. Comparisons of distances and angles between two lines and two planes in $R(4, 4)$ with both bases are shown in Table 6. While some of these formulas are identical, others have a factor of 2 when using the Witt basis that is not present when using the modified Witt basis.

3.5 Barycentric coordinates

Barycentric coordinates represent a point as an affine combination of the vertices of a simplex. The formulas for barycentric coordinates using the modified Witt basis are the same as those represented with the Witt basis; although powers of 2 appear in the computations, these powers of two cancel in the ratios of the formula. Table 7 gives these barycentric coordinate formulas.

4 Transformations in $R(4, 4)$

Goldman-Mann [2] used the rotors constructed from the generators $E_{ij}$, $F_{ij}$, $K_i$ of Doran et al. [1]:

$$E_{ij} = e_i e_j \pm \bar{e}_i \bar{e}_j$$

$$F_{ij} = e_i \bar{e}_j - \bar{e}_i e_j$$
Following the derivation of Goldman-Mann, we have

\[ e^{\theta e_i e_i} = \cosh(\theta) + \sinh(\theta)e_i \bar{e}_i \]

since \( (e_i \bar{e}_i)^2 = 1 \). Applying the rotor \( e^{\theta e_i e_i} \) to \( e_i \) and \( \bar{e}_i \),

\[
\begin{align*}
(cosh(\theta) - sinh(\theta)e_i \bar{e}_i) e_i (cosh(\theta) + sinh(\theta)e_i \bar{e}_i) &= (cosh^2(\theta) + sinh^2(\theta))e_i + 2sinh(\theta) cosh(\theta)\bar{e}_i \\
(cosh(\theta) - sinh(\theta)e_i \bar{e}_i) \bar{e}_i (cosh(\theta) + sinh(\theta)e_i \bar{e}_i) &= (cosh^2(\theta) + sinh^2(\theta))\bar{e}_i + 2sinh(\theta) cosh(\theta)e_i
\end{align*}
\]

Adding equations 3 and 4 and dividing by 2 yields

\[
\begin{align*}
(cosh(\theta) - sinh(\theta)e_i \bar{e}_i)w_i (cosh(\theta) + sinh(\theta)e_i \bar{e}_i) &= (cosh^2(\theta) + sinh^2(\theta) + 2 sinh(\theta) cosh(\theta))w_i \\
&= (cosh(2\theta) + sinh(2\theta))w_i = e^{2\theta}w_i
\end{align*}
\]
Table 8: Other nonsingular linear transformations on $R(4,4)$ with old basis and new basis

<table>
<thead>
<tr>
<th>Transformation</th>
<th>Witt Basis</th>
<th>Modified Witt Basis</th>
</tr>
</thead>
<tbody>
<tr>
<td>Reflection</td>
<td>$2w_i^* \land w_i$</td>
<td>$w_i^* \land w_i$</td>
</tr>
<tr>
<td>Classical shear</td>
<td>$w_i^*w_j$, $i \neq j$</td>
<td>$\frac{1}{2}w_i^*w_j$, $i \neq j$</td>
</tr>
</tbody>
</table>

while adding equations 3 and 4 and dividing by $\sqrt{2}$ yields

$$(\cosh(\theta) - \sinh(\theta)e_i\bar{e}_i)w_i(\cosh(\theta) + \sinh(\theta)e_i\bar{e}_i)$$

$$= (\cosh^2(\theta) + \sinh^2(\theta) + 2 \sinh(\theta) \cosh(\theta))w_i$$

$$= (\cosh(2\theta) + \sinh(2\theta))w_i = e^{2\theta}w_i$$

For $i \neq j$, $e_j$ commutes with $e_i\bar{e}_i$, so in either basis

$$(\cosh(\theta) - \sinh(\theta)e_i\bar{e}_i)w_j(\cosh(\theta) + \sinh(\theta)e_i\bar{e}_i) = w_j$$

Subtracting equations 3 and 4 and dividing by $\sqrt{2}$ gives the applications of the rotor $e^{\theta e_i\bar{e}_i}$ to elements in the space $W^*$:

$$(\cosh(\theta) - \sinh(\theta)e_i\bar{e}_i)w_i^*(\cosh(\theta) + \sinh(\theta)e_i\bar{e}_i)$$

$$= (\cosh(2\theta) - \sinh(2\theta))w_i^*$$

$$= (\cosh(-2\theta) - \sinh(-2\theta))w_i^* = e^{-2\theta}w_i^*.$$  

For $i \neq j$, $e_j$ commutes with $e_i\bar{e}_i$, so in either basis

$$(\cosh(\theta) - \sinh(\theta)e_i\bar{e}_i)w_j^*(\cosh(\theta) + \sinh(\theta)e_i\bar{e}_i) = w_j^*$$

Similar derivations show that the rotors generated by $E_{ij}$ and $F_{ij}$ operate in the same manner on both the $w_i, w_i^*$ basis and the $w_i, w_i^*$ basis, since in all three cases, the rotors have the same effect on the $e_i, \bar{e}_i$ basis, and our choice of $w$ or $w$ just results in combining the results on $e_i$ and $\bar{e}_i$ with a factor of $\frac{1}{2}$ or $\frac{1}{\sqrt{2}}$.

### 4.2 Other Nonsingular Linear Transformations in $R(4,4)$: Reflection and Classical Shear

Goldman-Mann [2] derived two nonsingular linear transformations for reflection and classical shear using the Witt basis; we rederive these formulas for the modified Witt basis. A comparison of these two sets of formulas is shown in Table 8. As examples of how the powers of 2 differ for the transformation formulas using the Witt basis and the modified Witt basis, we give a detailed derivation of reflection with the Witt basis and with the modified Witt basis.

### 4.3 Reflection: $2w_i^* \land w_i$ vs $w_i^* \land w_i$

We reproduce here a variation of the proof that the rotor $R_i = e_i\bar{e}_i$ is a reflection [3] to show an example of the use of the formula for reversing $w_i^*w_i$. In the Witt basis we have

$$e_i\bar{e}_i = (w_i + w_i^*)(w_i - w_i^*)$$

$$= w_i^*w_i - w_iw_i^* = 2w_i^* \land w_i$$

To see that $R_i$ is a reflection, we first show that

$$e_iw_i e_i = w_i^*, \quad e_iw_i^* e_i = w_i$$

$$\bar{e}_i w_i \bar{e}_i = w_i^*, \quad \bar{e}_i w_i^* \bar{e}_i = w_i.$$
Expanding the first of these equations,
\[ e_i w_i e_i = (w_i + w_i^*) w_i (w_i + w_i^*) = w_i^* w_i (w_i + w_i^*) \]
\[ = w_i^* w_i w_i^* = (1 - w_i w_i^*) w_i^* = w_i^*. \]
\[ (5) \]

The proofs for the other three equations are similar.

We now see that \( R_i = e_i \bar{e}_i \) is a reflection:
\[ R_i^{-1} w_i R_i = (-\bar{e}_i e_i) w_i (e_i \bar{e}_i) = -w_i \]
\[ R_i^{-1} w_i^* R_i = (-\bar{e}_i e_i) w_i^* (e_i \bar{e}_i) = -w_i^*. \]

Writing the rotor \( e_i \bar{e}_i \) in the modified Witt basis, we have
\[ e_i \bar{e}_i = \frac{\sqrt{2}(w_i + w_i^*) \sqrt{2}(w_i - w_i^*)}{2} \]
\[ = \frac{1}{2} (w_i^* w_i - w_i w_i^*) = w_i^* \wedge w_i \]

Thus, expanding \( e_i \bar{e}_i \) in the two variants of the Witt basis introduces different powers of 2.

The proof showing that \( e_i \bar{e}_i \) is a reflection in the modified Witt basis is essentially the same as the proof in the Witt basis. But the expansion of \( e_i w_i e_i \) is
\[ e_i w_i e_i = \frac{\sqrt{2}(w_i + w_i^*) w_i \sqrt{2}(w_i + w_i^*)}{2} = \frac{1}{2} w_i^* w_i (w_i + w_i^*) \]
\[ = \frac{1}{2} w_i^* w_i w_i^* = \frac{1}{2} (2 - w_i w_i^*) w_i^* = w_i^*. \]
\[ (6) \]

Comparing Equation 5 to Equation 6 gives an example of the two variations of the formula for swapping \( w_i, w_i^* \); in the example in these two equations, the powers of two cancel in the modified Witt basis, yielding the same formula for reflection when the rotor \( R_i \) is written as the geometric product of \( e_i, \bar{e}_i \), although when expressing \( R_i \) as the outer product of \( w_i^*, w_i \), a factor of 2 appears when using the Witt basis.

4.4 Transformations relative to \( w, w^* \)
Goldman-Mann [2] rewrite rotors to get an expression in terms of the basis \( w_i, w_j, w_i^*, w_j^* \). The derivations using the modified Witt basis are essentially the same the derivations using the Witt basis in Goldman and Mann’s paper; Tables 9 and 10 give a comparison of transformations using the Witt basis \( w_i, w_j, w_i^*, w_j^* \) and the modified Witt basis \( w_i, w_j, w_i^*, w_j^* \). With the exception of shear, the formulas in the modified Witt basis are simpler (do not have factors of 2) than those in the Witt basis.

5 Conclusions
In this paper, gave a variation on an earlier \( R(4, 4) \) model [2, 3]. The only difference between our variation and the earlier model of \( R(4, 4) \) is in a scaling factor for one of the bases. With this new basis, the coefficients in the geometric formulas are mostly unity (as compared to powers of 2 when using the original basis). In addition, the coefficients in the transformation formulas are also simpler.

Acknowledgements
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Table 9: Blades and the corresponding rotors and transformations on $R^3$ based on classical shear in $R^4$ with old and new bases

<table>
<thead>
<tr>
<th>Blade</th>
<th>w-Rotor</th>
<th>$w$-Rotor</th>
<th>Transformation</th>
</tr>
</thead>
</table>
| $w_i^* \wedge w_j$ $i \neq j$ | $1 + tw_i^* \wedge w_j$  | $1 + \frac{1}{2} tw_i^* \wedge w_j$ | Classical shear in the $w_iw_j$-plane  
(Only $w_i$-direction changes) |
| $w_i^* \wedge w_j$ $j \neq 0$ | $1 + tw_i^* \wedge w_j$  | $1 + \frac{1}{2} tw_i^* \wedge w_j$ | Classical shear in the $w_iw_j$-plane  
(Only $w_i$-direction changes) |
| $w_j^* \wedge w_0$ $j \neq 0$ | $1 - w_j^* \wedge w_0$  | $1 - \frac{1}{2} w_j^* \wedge w_0$ | Pseudo-perspective normal to $w_j$-direction |
| $w_i^* \wedge w_i$ $i \neq 0$ | $c + 2sw_i^* \wedge w_i$ | $c + sw_i^* \wedge w_i$ | Nonuniform scaling by $e^{2\theta}$ in the $w_i$-direction |
| $w_0^* \wedge w_0$ | $c + 2sw_0^* \wedge w_0$ | $c + sw_0^* \wedge w_0$ | Uniform scaling of points by $e^{-2\theta}$ |

Table 10: Rotors for rotation and scissors shear ($s = \sin(\theta)$, $c = \cos(\theta)$); Scissor Shear Generator: $e_i \bar{e}_j - \bar{e}_i e_j$.
Rotation Generator: $e_i e_j - \bar{e}_i \bar{e}_j$.

<table>
<thead>
<tr>
<th>Transformation</th>
<th>w-Rotor</th>
<th>w-Rotor</th>
</tr>
</thead>
<tbody>
<tr>
<td>Rotation by $2\theta$ in $w_iw_j$-plane</td>
<td>$c^2 + 2sc(w_i^* \wedge w_j - w_j^* \wedge w_i)$</td>
<td>$c^2 + sc(w_i^* \wedge w_j - w_j^* \wedge w_i)$</td>
</tr>
<tr>
<td></td>
<td>$+4s^2(w_i^* \wedge w_i)(w_j^* \wedge w_i)$</td>
<td>$+s^2(w_i^* \wedge w_i)(w_j^* \wedge w_i)$</td>
</tr>
<tr>
<td>Scissors shear in $w_iw_j$-plane</td>
<td>$c^2 + 2sc(w_i^* \wedge w_j + w_j^* \wedge w_i)$</td>
<td>$c^2 + sc(w_i^* \wedge w_j + w_j^* \wedge w_i)$</td>
</tr>
<tr>
<td></td>
<td>$-4s^2(w_i^* \wedge w_i)(w_j^* \wedge w_i)$</td>
<td>$-s^2(w_i^* \wedge w_i)(w_j^* \wedge w_i)$</td>
</tr>
</tbody>
</table>
References


