Linear-Time Algorithms for Proportional Contact Graph Representations*

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Abstract. In a proportional contact representation of a planar graph, each vertex is represented by a simple polygon with area proportional to a given weight, and edges are represented by adjacencies between the corresponding pairs of polygons. In this paper we study proportional contact representations that use rectilinear polygons without wasted areas (white space). In this setting, the best known algorithm for proportional contact representation of a maximal planar graph uses 12-sided rectilinear polygons and takes $O(n \log n)$ time. We describe a new algorithm that guarantees 10-sided rectilinear polygons and runs in $O(n)$ time. We also describe a linear-time algorithm for proportional contact representation of planar 3-trees with 8-sided rectilinear polygons and show that this optimal, as there exist planar 3-trees that requires 8-sided polygons. Finally, we show that a maximal outer-planar graph admits a proportional contact representation with 6-sided rectilinear polygons when the outer-boundary is a rectangle and with 4 sides otherwise.

1 Introduction

Representing planar graphs as *contact graphs* has been a subject of study for many decades. In such a representation, vertices correspond to geometrical objects, such as line-segments or polygons, while edges correspond to two objects touching in some pre-specified fashion. In this paper, we consider *side contact representations* of planar graphs, where vertices are simple polygons, and adjacencies are non-trivial side-contacts between the corresponding polygons. In the weighted version of the problem, the goal is to find a contact representation of G where the area of the polygon for each vertex is proportional to the weight of the vertex, which is given in advance. We call such a representation a *proportional contact representation* of G. Such representations often lead to a more compelling visualization of a planar graph than usual node-link representations [4] and have practical applications in cartography, VLSI Layout, and floor-planning. Rectilinear polygons with small number of sides (or corners) are often desirable due to esthetic, practical, and cognitive requirements. In VLSI design and architectural floor-planning, it is also desirable to minimize the unused area in the

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representation. Hence we address the problem of constructing a proportional contact representation of a planar graph with rectilinear polygons with few sides, so that the representation contains no unused area.

1.1 Related Work

Contact representations of planar graphs can be dated back to 1936 when Koebe showed that any planar graph has a representation by touching circles. While touching circles or touching triangles provide point-contact representations, side-contact representations have also been considered. For example, Gansner *et al.* [6] show that 6-sided polygons are sometimes necessary and always sufficient for side-contact representation of any planar graph with convex polygons.

Applications in VLSI or architectural layout design encourage the use of rectilinear polygons in a contact representation that fills a rectangle. In this setting it is known that 8 sides are sometimes necessary and always sufficient [7, 11, 21]. A characterization of the graphs admitting a more restricted rectangle-representation is given by Koźmiński and Kinnen [10] and in the dual setting by Ungar [19]. A similar characterization of graphs having representations with 6-sided rectilinear polygons is given by Sun and Sarrafzadeh [17]. Buchsbaum *et al.* [4] give an overview on the state of the art concerning rectangle contact graphs.

In the results summarized above, the vertex weights and polygonal areas are not considered. The weighted version of the problem, that of *proportional contact representations* has applications in *cartograms*, or value-by-area maps. Here, the goal is to redraw an existing geographic map so that a given weight function (e.g., population) is represented by the area of each country. Algorithms by van Kreveld and Speckmann [20] and Heilmann *et al.* [8] yield representations with rectangular polygons, but the adjacencies may be disturbed. De Berg *et al.* describe an adjacency-preserving algorithm for proportional contact representation with at most 40 sides for an internally triangulated plane graph G [5]. This was later improved to 34 sides [9].

The problem has also been studied in the dual settings, where the weights are assigned to the internal faces of a plane graph (instead of the vertices). All planar cubic graphs admit such a drawing [18] as do all planar partial 3-trees [2], but not all planar graphs [14]. Proportional rectilinear drawings with 8-sided polygons can be found for special classes of planar graphs [13], but this approach does not extend to general planar graphs. In a recent paper, Biedl and Velázquez [3] describe the best general result, with an $O(n \log n)$ algorithm for proportional representation of cubic triconnected graphs with 12-sided rectilinear polygons. Since the dual of a maximal planar graph is a cubic triconnected graph, this result yields a proportional contact representation of a maximal planar graph with 12-sided rectilinear polygons.

1.2 Our Contribution

Our main contribution is an improvement from the $O(n \log n)$ algorithm for 12-sided rectilinear polygons [3], with a new algorithm based on Schnyder realizers that runs in $O(n)$ time and provides a proportional contact representation of a maximal planar graph with 10-sided polygons.

We also describe a linear-time algorithm for proportional contact representations of planar 3-trees with 8-sided rectilinear polygons and show that this optimal, as there exist planar 3-trees that require 8-sided polygons. Finally, we show that a maximal outer-planar graph admits a proportional contact representation with 6-sided rectilinear polygons when the outer-boundary is a rectangle and with 4 sides otherwise.

2 Representations for Maximal Planar Graphs

Here we describe the algorithm for 10-sided rectilinear polygons. We construct the proportional representation of a maximally planar graph using Schnyder realizers [16], which we review briefly. A *Schnyder realizer* of a fully triangulated graph G is a partition of the interior edges of G into three sets T_1, T_2 and T_3 of edges that can be directed such that for each interior vertex v (1) v has out-degree exactly one in each of T_1, T_2 and T_3, and (2) the counterclockwise order of the edges incident to v is: entering T_1, leaving T_2, entering T_3, leaving T_1, entering T_2, leaving T_3.

The first condition implies that each T_i, $i = 1, 2, 3$ defines a tree spanning all the interior vertices and rooted at exactly one exterior vertex such that the edges are directed towards the root. Schnyder proved that any triangulated planar graph has such a realizer and it can be computed in $O(n)$ time [16].

Theorem 1. *Let $G = (V, E)$ be a maximal planar graph and let $w : V \rightarrow R^+$ be a weight function. Then a proportional contact representation Γ with respect to w can be constructed in linear time such that each vertex of G is represented by a 10-sided rectilinear polygon in Γ, and there is no wasted area.*

We prove Theorem 1 by giving a linear-time algorithm to construct such a representation Γ of G, where each vertex of G is represented by a 10-sided rectilinear polygon with a fixed shape, illustrated in Figure 1. (Some sides of the polygon may be degenerate.) This polygon can be decomposed into four rectangles called *foot*, *leg*, *bridge* and *body* of the polygon. The region bound by the parallel horizontal lines containing the top and the bottom of the bridge is the *bridge-strip*, and the *foot-strip* is defined analogously.

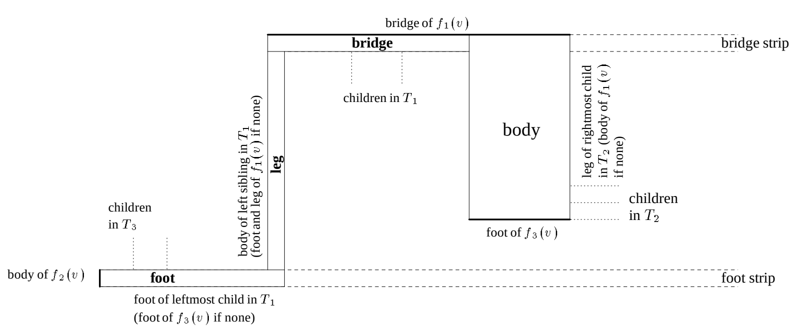

Fig. 1. A 10-sided rectilinear polygon with decomposition into foot, leg, bridge and body.

Let $G = (V, E)$ be a maximal plane graph with the three outer vertices v_1, v_2 and v_3 in counterclockwise order, and let $w : V \to R^+$ be a weight function. We first find a Schnyder realizer of G that partitions the interior edges into three rooted trees T_1, T_2 and T_3 rooted at v_1, v_2 and v_3, and with all their edges oriented towards the roots of the trees. We add the external edges (v_1, v_2), (v_1, v_3) to T_1 and (v_2, v_3) to T_2, so that all the edges of G are partitioned into the three trees. For each vertex v of G, let $f_i(v)$, $i = 1, 2, 3$ be the parent of v in T_i.

Let R be a rectangle with area equal to $\sum_{v \in V} w(v)$. We construct a proportional contact representation of G inside R. We start by cutting a rectangle $P(v_1)$ with area $w(v_1)$ for v_1 from the top of R and cutting a rectangle $P(v_2)$ with area $w(v_2)$ for v_2 from the left side of $R - P(v_1)$. In the remaining part $R' = R - P(v_1) - P(v_2)$ of the rectangle, we draw the polygons for the other vertices.

The main idea of the algorithm is to draw the polygons such that for each vertex v of G, the edges $(v, f_i(v))$ are realized as follows: The top of the bridge of $P(v)$ is adjacent to the bottom of the bridge of $P(f_1(v))$, the left of the foot of $P(v)$ is adjacent to the right of the body of $P(f_2(v))$ and the bottom of the body of $P(v)$ is adjacent to the top of the foot of $P(f_3(v))$. See also Figure 1. Note that if we ensure those adjacencies, then there cannot be any other adjacencies since graph G is maximal planar, and correctness follows.

The other crucial idea is that the bridge and foot have small height and the leg has very small width, so that they together occupy less area than the weight of v. (Ensuring that their width/height is no more than $w(v)/(W + H)$ if R' is a $W \times H$-rectangle will do.) The bulk of the weight for v is hence in the body of v.

Our algorithm visits vertices in depth-first order in the tree T_1 and builds $P(v)$ partially before and partially after visiting the children of v (in left-to-right order according to the planar embedding.) Thus the visit at v has the following steps:

1. Fix the bridge-strip, foot and leg of $P(v)$;
2. For each child u of v in T_1 in left-to-right order, call the algorithm recursively for u;
3. Fix the bridge and the body of $P(v)$;
4. Set the foot strips for children of v in T_2.

Details of Step 4: We explain step 4 first, since it is vital for the other steps. Any vertex u that is a child of v in T_2 can be shown to come after v in the left-to-right depth-first search order of T_1. To ensure that all these vertices can attach to the right side of the body of $P(v)$ without gaps, we reserve a horizontal strip of small height for the foot of $P(u)$ at the right side of the body of $P(v)$. These foot-strips are placed starting at the bottom of the body of $P(v)$, all adjacent to each other, and assigned according to the counter-clockwise order of edges around v. Choosing the strips small enough ensures that they will all fit along the body. We presume that this operation has also been applied when we place $P(v_2)$, so that for every vertex $v \neq v_1, v_2$, the foot strip of v is already set when we visit v.

Details of Step 1: When we visit vertex v, the foot-strip of $P(v)$ hence has been set. The bridge-strip of $P(f_1(v))$ has also been fixed since $f_1(v)$ is visited before v. We fix the bridge strip of $P(v)$ with small height just under the bridge-strip of $P(f_1(v))$ so that the two bridge strips touch each other.

To fix foot and leg of v, we have two cases. If $f_2(v)$ is not the left sibling of v in T_1, then the foot extends (in the foot strip of v) until the body of the leftmost child of v in T_3, or until the leg of $f_1(v)$ if there is no such child. The leg then extends upwards until the bridge strip of v. If $f_2(v)$ is the left sibling of v in T_1, then the foot of v vanishes; we extend the leg of v from the bottom of the footstrip for v until the bottom of the bridge strip of v. Note that in either case all parts of neighbours required for this drawing have been placed already.

Details of Step 3: To fix the bridge of $P(v)$, there are again two cases. If v has children in T_1, then we complete the bridge of $P(v)$ so that it extends from the leg of $P(v)$ to the rightmost side of any child of v in T_1 (they all have been drawn already.) If v has no children in T_1, then the bridge vanishes and the body of v contains the leg of v. Then we fix the body of $P(v)$ so that it extends from the bottom of the bridge-strip of $P(f_1(v))$ to the foot-strip of $P(f_3(v))$. This is possible because $f_2(f_3(v))$ always precedes v in the traversal of T_1, and so the foot-strip of $P(f_3(v))$ has already been fixed. After knowing exactly the area consumed by foot, leg and bridge of $P(v)$, as well as the height of the body, we can choose the width so that the area of $P(v)$ is $w(v)$. (Since the foot, leg and bridge together consume little area, the width of the body is positive.)

Figure 2(b) illustrates a proportional contact representation of the maximal planar graph in Figure 2(a) computed with our algorithm. A more detailed step-by-step correction and details of the proof of correctness is given in the appendix.

The linear-time implementation consists of computing a Schnyder realizer of G [16], and the traversal of tree T_1 together with the local computations of the different parts of the polygons. $\square$

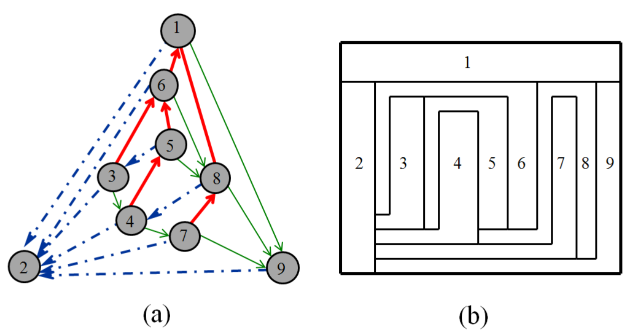

Fig. 2. (a) A maximal planar graph G, (b) a proportional contact representation of G.

So we have now established that 10 sides are sufficient for proportional contact representation with rectilinear polygons. Yeap and Sarrafzadeh [21] gave an example of a maximal planar graph, which is also a planar 3-tree, for which at least 8-sided polygons are necessary. In very recent work [1] we managed to prove that 8-sided polygons are also sufficient. However, in contrast to the 10-gon construction given above, the proof of this result is not constructive, and the representation can be found only via numerical approximation. So while the construction with 10-gons is not theoretically best possible, it is probably of higher interest for practical settings.

3 Representations of Planar 3-trees

Here we describe proportional contact representations of planar 3-trees with fewer sides (8) in each polygon. A *3-tree* is either a 3-cycle or a graph G with a vertex v of degree three in G such that $G - v$ is a 3-tree and the neighbors of v are adjacent. If G is planar, then it is called a *planar 3-tree*. A *plane 3-tree* is a planar embedding of a planar 3-tree. It is easy to see that starting with a 3-cycle, any planar 3-tree can be formed by recursively inserting a vertex inside a face and adding an edge between the newly added vertex and each of the three vertices on the face [2, 12].

Using this simple construction, we can create in linear time a *representation tree* for G, which is an ordered rooted ternary tree T_G spanning all the internal vertices of G. The root of T_G is the first vertex we have to insert into the face of the three outer vertices. Adding a new vertex v in G will introduce three new faces belonging to v. The first vertex w we add in each of these faces will be a child of v in T_G. The correct order of T_G can be obtained by adding new vertices according to the counterclockwise order of the introduced faces. For any vertex v of T_G, we denote by U_v, the set of the descendants of v in T_G including v. The *predecessors* of v are the neighbors of v in G that are not in U_v. Clearly each vertex of T_G has exactly three predecessors. We now have the following lemma.

Lemma 1. *Let $G = (V, E)$ be a plane 3-tree and let $w : V \to R^+$ be a weight function. Then a proportional contact representation of G can be obtained in linear time where each vertex of G is represented by an 8-sided rectilinear polygon.*

Proof. Let T_G be the representation tree of G. For any vertex v of T_G, let $W(v)$ denote the summation of the weights assigned to each of the vertices in U_v. A linear-time bottom-up traversal of T_G is sufficient to compute $W(v)$ for each vertex v of G. In the followings, we construct a proportional contact representation of G inside any rectangle R with area equal to the summation of the weights for all the vertices of G.

Let a, b, c be the three outer vertices of G in the counterclockwise order. We first draw the polygons for a, b and c. We cut a rectangle $P(a)$ with area $w(a)$ for a form the top of R, cut a rectangle $P(b)$ with area $w(b)$ from the left side of $R - P(a)$ and cut an L-shaped strip $P(c)$ of area $w(c)$ for c from the right side and the bottom of $R - P(a) - P(b)$, as illustrated in Figure 3(a). We now draw the polygons for the vertices in T_G inside the rectangle $R - P(a) - P(b) - P(c)$ by a top-down traversal of T_G. While we traverse a vertex v of T_G, we recursively draw the polygons for the vertices of U_v inside a rectangle R_v with area $W(v)$ such that R_v shares two of its sides with the polygon for one of the predecessors of v and the other two sides with the polygons for the other two predecessors. Note that this condition holds for the rectangle $R - P(a) - P(b) - P(c)$ representing the root of T_G. Let v be a vertex of T_G with predecessors p_1, p_2, p_3 and let $pqrs$ be the rectangle with area $W(v)$ where ps, pq and qrs are part of the boundary of the polygons for p_1, p_2 and p_3, respectively. From $pqrs$, we then cut three rectangles $R_1 = t_1t_2rs$, $R_2 = pt_3t_4t_5$ and $R_3 = qt_6t_7t_8$ with areas $W(u_1)$, $W(u_2)$ and $W(u_3)$, respectively, as illustrated in Figure 3(b), where u_1,

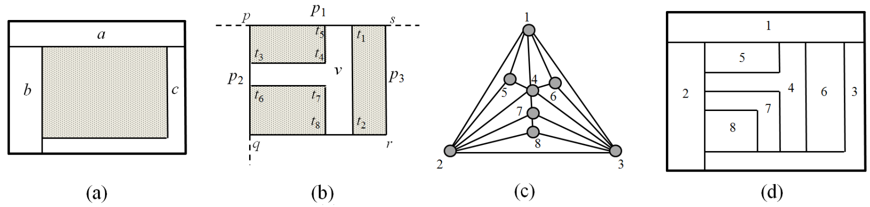

Fig. 3. (a)–(b) Illustration for the proof of Lemma 1, (c) a planar 3-tree G, and (d) a proportional contact representation of G.

u_2 and u_3 are the three children of v in T_G (some of them might be empty). Then the 8-sided polygon obtained by $pqrs - R_1 - R_2 - R_3$ has area $w(v)$ and has common boundary with all of the polygons representing its predecessors. Finally, we recursively fill out the rectangles R_1, R_2, R_3 by polygons representing the vertices in U_{u_1}, U_{u_2} and U_{u_3}, respectively. Clearly the polygon representing each vertex v of G can be computed in constant time. Thus the time complexity for constructing the representation of G is linear. □

Figure 3(d) illustrates a proportional contact representation of the planar 3-tree in Figure 3(c) computed by the algorithm described above. The algorithmic upper bound of 8 sides per polygon is also matched with the corresponding lower bound with a planar 3-tree for which at least 8-sided polygons are necessary in a proportional contact representation with rectilinear polygons [21]. We thus have the following result.

Theorem 2. *Polygons with 8 sides are always sufficient and sometimes necessary for proportional contact representations of planar 3-trees with rectilinear polygons.*

4 Representations for Maximal Outer-planar Graphs

Here we describe proportional contact representations of maximal outer-planar graphs with even fewer sides (6 and 4) in each polygon. An *outer-planar graph* is a graph that has an *outer-planar embedding*, i.e., a planar embedding with every vertex in the outer face. An outer-planar graph to which no edges can be added without violating outer-planarity is a *maximal outer-planar graph*. It is easy to see that each internal face in an outer-planar embedding of a maximal outer-planar graph is a triangle, and for $n \geq 3$ the outer-face is a simple cycle containing all vertices. We will give a linear-time algorithm to construct a proportional contact representation of a maximal outer-planar graph with rectangles. Before that, we need the following definitions.

Let Γ be a contact representation using rectangles for vertices (but with the outside not necessarily a rectangle). Let B be the bounding box of Γ. We say that a vertex v *occupies the top* of a representation Γ if there exists a horizontal line ℓ such that the rectangle representing v is exactly the intersection of B with the upper half-space of ℓ. In other words, the rectangle of v contains all of the top end of the bounding box of Γ. Similarly we define that a vertex v *occupies the right* of Γ.

Lemma 2. *Let G be a maximal outer-planar graph, and let (s, t) be an edge on the outer-face, with s before t in clockwise order. Then a proportional contact-representation Γ of G with rectangles can be computed in linear time such that s occupies the top of Γ and t occupies the right of $\Gamma - s$.*

Proof. We give an algorithm that recursively computes Γ. Constructing Γ is easy when G is a single edge (s, t); see Fig. 4. We thus assume that G has at least 3 vertices. Let x be the (unique) third vertex on the inner face that is adjacent to (s, t). Then graph G can be split into two graphs at vertex x and edge (s, t): $G[s, x]$ consists of the graph induced by all vertices between s and x in counter-clockwise order around the outer-face, and $G[x, t]$ consists of the graph induced by the vertices between t and x.

Recursively draw $G[s, x]$ and remove s from it; call the result Γ_s. Recursively draw $G[x, t]$ and remove x and t from it; call the result Γ_t. Then scale the width of Γ_t until the bounding box of Γ_t is less wide than the rectangle of x in Γ_s. To maintain a proportional contact representation, scale the height of Γ_t by the inverse of the scale-factor for the width. Now Γ_t can be attached at the bottom right end of the representation of x in Γ_s. Add a rectangle for t on the right that spans the whole height (and extends below it at the bottom), and make its width such that its area is as prescribed for t. Add a rectangle for s such that it spans the whole width (and extends below it at the left), and make its height such that its area is as prescribed for s. This gives the desired representation.

We now show that the above algorithm can be implemented in linear time. In order to do this, we make sure that all coordinates in the representation are scaled at most once. Let T be the dual graph of G minus the vertex for the outer-face; it is easy to see that T is a tree with maximum degree three. Root T at the vertex that corresponds to the inner face $\{s, x, t\}$; then the subtrees of T correspond to the dual trees of the subgraphs. Rather than re-scaling Γ_t at each recursive step, we only re-scale the bounding box of Γ_t and store at the node of T that represents $G[t, x]$ the scale-factors for the width and height that must be applied to all nodes in Γ_t. At the end of the algorithm a linear-time top-down traversal finds the scaling factor for each vertex v of T by multiplying all the scaling factors stored along the path from v_x to v. Then with another linear-time top-down traversal of T we can compute the coordinates of all the points in Γ, which concludes the construction. □

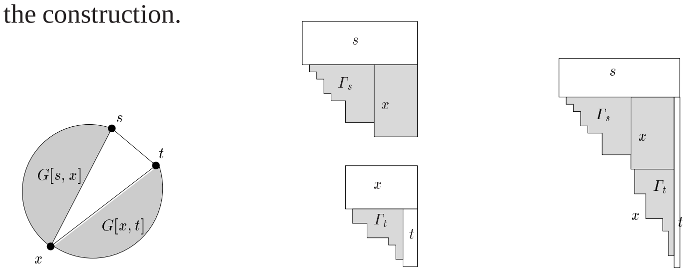

Fig. 4. Combining the drawings of two subgraphs.

Since a rectangle is a rectilinear polygon with the fewest sides possible, the representation obtained by the above algorithm is also optimal. However, the outer boundary of the representation obtained by our construction, has size $\Theta(n)$. It was already known that the outer-face cannot be a rectangle if the vertices are rectangles [15], but we improve this to a stronger result:

Lemma 3. *There exists a maximal outer-planar graph for which any contact representation with rectangles requires $\Omega(n)$ sides on the outer-face.*

Proof. Consider any maximal outer-planar graph G such that $\lfloor n/2 \rfloor$ vertices have degree two (any maximal outer-planar graph whose inner dual is a full binary tree suffices). Suppose Γ is a proportional contact representation of G with rectangles. Since rectangles are convex, no two of them can share two sides. Therefore any vertex v of degree 2 shares at most two of its sides with other vertices, and so at least two of its sides with the outer boundary of Γ. Furthermore, these two sides must be consecutive on $P(v)$, since otherwise v would be a cut vertex in G. The common endpoint of these two sides is then a corner of the outer boundary of Γ, so the outer-face has at least $\lfloor n/2 \rfloor$ sides. □

Lemma 3 implies that there exists outer-planar graphs for which any contact representation with an the outer-boundary of constant size requires at least one of the polygons to have at least six sides. With the following lemma we show that this lower bound of six sides can also be matched with any given weights.

Lemma 4. *Let $G = (V, E)$ be a maximal outer-planar graph and let $w : V \to R^+$ be a weight function. Then a proportional contact-representation Γ of G with 6-sided rectilinear polygons can be computed in linear time such that the outer-boundary of Γ is a rectangle.*

Proof. It is quite straightforward to prove this by analyzing the structure of an outer-planar graph, but it also follows from both previous results in this paper. We only sketch this here.

First, if G is maximal outer-planar, then we can add one vertex v_0 to it that is adjacent to all others. Then create a Schnyder realizer such that v_0 is the root of tree T_1. Then all its incident edges are labeled with 1, which means that all other vertices are leaves in tree T_1. Apply our construction from Section 2. One can easily verify that vertices that are leaves in T_1 are drawn with 6-gons in this construction. Omitting the added vertex v_0 (which is a rectangle that spans the top) yields the desired representation.

As a second proof, observe that $G \cup v_0$ is also a 3-tree, and moreover, any vertex v has at most two children in T_G. Apply the construction of Section 3 and split the rectangle of weight $W(v)$ in such a way that $P(v)$ has at most 6 sides; one can verify that this is always possible if v has at most two children in T_G. □

Summing up all the results in this section, we have the following theorem.

Theorem 3. *For a rectilinear proportional contact representation of a maximal outer-planar graph, rectangles are always sufficient and necessary, and six-sided polygons are sometimes necessary (and always sufficient) when the outer-boundary has a constant number of sides.*

5 Representations for Maximal Series-Parallel Graphs

In this section, we prove that maximal series-parallel graphs have proportional contact representations with 6 sides and a rectangle as outer-face boundary. A *series-parallel* graph is a graph G that has two terminals s and t, and either G is an edge (s, t), or it has been obtained with one of the following two operations: (1) (Parallel combination) G consists of two or more series-parallel graphs that all have the terminals s and t. (2)

(Combination in series) G consists of two series-parallel graphs, one with terminals s and some other vertex x, and the other with terminals x and t. As usual, a maximal series-parallel graph is a series-parallel graph to which we cannot add any more edges and maintain a simple series-parallel graph. It is well-known that any series-parallel graph is planar and that these are the same as the partial 2-trees.

One can easily show that a maximal series-parallel graph has $2n - 3$ edges, and hence it cannot be internally triangulated unless all vertices are on the outer-face (in case of which it is outer-planar.) So in order to create proportional contact representations, we must allow holes. However, we will show that these holes can be made arbitrarily small while using only 6-sided rectilinear polygons.

To state our result precisely, we use the following notation. For any vertex set V', use $w(V')$ to denote the sum of weights of vertices in V'. Now we have:

Lemma 5. *Let G be a maximal series-parallel graph with terminals s, t and let $w : V \to R^+$ be a weight-function. Let $\varepsilon > 0$ be arbitrarily small. Let $\mathcal{R}$ be any rectangle of area $w(V - \{s,t\}) + \varepsilon$. Then $G - \{s,t\}$ has a proportional contact representation inside $\mathcal{R}$ such that a vertex v touches the top/right side of $\mathcal{R}$ if and only if v is adjacent to s/t in G.*

Proof. We prove this by induction on the number of vertices. In the base case, G consists of edge (s,t) only, and the claim is vacuously true since $G - \{s,t\}$ is empty. So now assume that G has at least 3 vertices. Since G is a maximal series-parallel graph, edge (s,t) must exist. Therefore G must be obtained in a parallel combination of subgraphs $G_0, G_1, \ldots, G_k$, all with terminals s and t. (We presume the naming is such that G_0 is the edge (s,t).) We make k as large as possible, i.e., each subgraph G_i for $i > 0$ was obtained in a combination in series of subgraphs G_i^s and G_i^t, where G_i^s has terminals s and x_i and G_i^t has terminals x_i and t. The idea is to assign rectangles to each of these subgraphs G_i^α (for $i = 1, \ldots, k$ and $\alpha \in \{s,t\}$) and place the drawings inside $\mathcal{R}$ suitably. Let $V_i^\alpha = V(G_i^\alpha) - \{x_i, \alpha\}$. We proceed as follows:

1. First, remove a (very slim) rectangle adjacent that spans the left side of $\mathcal{R}$ and has area $\varepsilon' := \varepsilon/(5k+2)$.[5]
2. From the rectangle that remains, remove a very slim rectangle of area ε' that spans the bottom.
3. From the rectangle that remains, remove an L-shaped 6-sided polygon $P(x_k)$ that spans the bottom and the left side. Choose the side-lengths such that $P(x_k)$ has area $w(x_k)$.
4. From the rectangle that remains, remove a rectangle that spans the left side. Choose its width such that its area is $w(V_k^s) + 2\varepsilon'$. Then split it horizontally so that the rectangle below has area ε' while the rectangle R_k^s above has area $w(V_k^s) + \varepsilon'$.
5. From the rectangle that remains, remove a very slim rectangle of area ε' that spans the left side.
6. From the rectangle that remains, remove a rectangle R_k^t that spans the bottom. Choose its width such that its area is $w(V_k^t) + \varepsilon'$.

[5] Any distribution of ε area among the empty region and the rectangles is feasible, as long as they are all non-zero.

7. Repeat steps 3-6 for $k-1, k-2, k-3, \ldots, 1$.
8. By choice of ε' and the areas for all rectangles and L-shapes, all that remains of $\mathcal{R}$ after removing rectangle R_1^t is a slim rectangle (adjacent to the top of R_1^t) of area ε'.

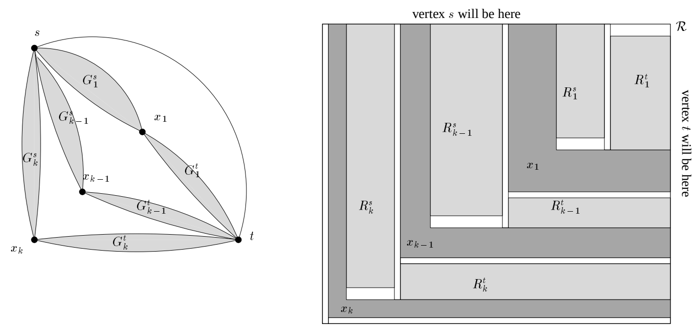

Fig. 5. The construction for a series-parallel graph. $k = 3$ in this example.

Figure 5 illustrates the construction. Note that for each rectangle R_i^α, two sides are adjacent to empty space, one side is adjacent to x_i, and the other side is adjacent to the boundary of $\mathcal{R}$ where terminal α will be located. Furthermore, R_i^α has weight $w(V_i^\alpha) + \varepsilon'$. Hence we can call the algorithm recursively for G_i^α, using rectangle R_i^α and ε'. The resulting contact representation of $G_i^\alpha - \{x_i, \alpha\}$ can then be placed (after rotating/flipping as needed to make terminals match up) inside R_i^α. This yields the desired proportional contact representation for $G - \{s, t\}$. $\square$

We have thus shown that 6 sides are sufficient for series-parallel graphs. To see that they are necessary, consider $K_{2,4}$, which is a series-parallel graph. No matter what embedding we choose, there will always be a vertex that is enclosed by a triangle in $K_{2,4}$. Since three rectangles cannot enclose a non-zero area, this shows that $K_{2,4}$ requires 6 sides in any contact representation. We hence have the following theorem:

Theorem 4. *6 sides are always sufficient and sometimes necessary for a proportional contact representation of maximal series-parallel graphs with arbitrarily small holes.*

6 Conclusion

We gave an algorithm for a proportional contact representation of a maximal planar graph with 10-sided rectilinear polygons, which improves on the previously known upper bound of 12.

We also described algorithms for special classes of planar graphs with 8-sided rectilinear polygons and a similar approach might be extended to general planar graphs. Finally, we described algorithms for 6-sided and 4-sided rectilinear representation for outer-planar graphs.

All algorithms in this paper can be implemented in linear time, and require nothing more complicated than Schnyder realizers and other elementary planar graph manipulations. In contrast, the very recent improvement in the number of sides to 8 [1], the proof is non-constructive and requires numerical approximations to find the contact representation. Finding a constructive proof (and preferably linear-time algorithm) to construct 8-sided proportional contact representations of maximal planar graphs remains open.

References

1. M. J. Alam, T. Biedl, S. Felsner, A. Gerstner, M. Kaufmann, S. Kobourov, and T. Ueckert. Proportional contact graph representations with optimal complexity. In preparation.
2. T. Biedl and L. E. Ruiz Velázquez. Drawing planar 3-trees with given face-areas. In *Proc. 17th International Symposium on Graph Drawing (GD'09)*, pages 316–322, 2010.
3. T. Biedl and L. E. Ruiz Velázquez. Orthogonal cartograms with few corners per face. In *12th Algorithms and Data Structures Symposium (WADS)*, 2011. To appear.
4. A. L. Buchsbaum, E. R. Gansner, C. M. Procopiuc, and S. Venkatasubramanian. Rectangular layouts and contact graphs. *ACM Transactions on Algorithms*, 4(1), 2008.
5. M. de Berg, E. Mumford, and B. Speckmann. On rectilinear duals for vertex-weighted plane graphs. *Discrete Mathematics*, 309(7):1794–1812, 2009.
6. E. R. Gansner, Y. Hu, M. Kaufmann, and S. G. Kobourov. Optimal polygonal representation of planar graphs. In *9th Latin Am. Symp. on Th. Informatics (LATIN)*, pages 417–432, 2010.
7. X. He. On floor-plan of plane graphs. *SIAM Journal of Computing*, 28(6):2150–2167, 1999.
8. R. Heilmann, D. A. Keim, C. Panse, and M. Sips. Recmap: Rectangular map approximations. In *10th IEEE Symp. on Information Visualization (InfoVis 2004)*, pages 33–40, 2004.
9. A. Kawaguchi and H. Nagamochi. Orthogonal drawings for plane graphs with specified face areas. In *4th Conf. on Theory and Applications of Models of Comp.*, pages 584–594, 2007.
10. K. Koźmiński and E. Kinnen. Rectangular duals of planar graphs. *Networks*, 15:145–157, 1985.
11. C.-C. Liao, H.-I. Lu, and H.-C. Yen. Compact floor-planning via orderly spanning trees. *Journal of Algorithms*, 48:441–451, 2003.
12. D. Mondal, R. I. Nishat, M. S. Rahman, and M. J. Alam. Minimum-area drawings of plane 3-trees. In *CCCG*, pages 191–194, 2010.
13. M. S. Rahman, K. Miura, and T. Nishizeki. Octagonal drawings of plane graphs with prescribed face areas. *Computational Geometry*, 42(3):214–230, 2009.
14. G. Ringel. Equiareal graphs. In R. Bodendiek, editor, *Contemporary Methods in Graph Theory*, pages 503–505. Wissenschaftsverlag, 1990.
15. I. Rinsma. Nonexistence of a certain rectangular floorplan with specified area and adjacency. *Environment and Planning B: Planning and Design*, 14:163–166, 1987.
16. W. Schnyder. Embedding planar graphs on the grid. In *Proceedings of the 1st ACM-SIAM Symposium on Discrete Algorithms (SODA)*, pages 138–148, 1990.
17. Y. Sun and M. Sarrafzadeh. Floorplanning by graph dualization: L-shaped modules. *Algorithmica*, 10(6):429–456, 1993.
18. C. Thomassen. Plane cubic graphs with prescribed face areas. *Combinatorics, Probability & Computing*, 1:371–381, 1992.
19. P. Ungar. On diagrams representing graphs. *J. London Math. Soc.*, 28:336–342, 1953.
20. M. J. van Kreveld and B. Speckmann. On rectangular cartograms. *Computational Geometry*, 37(3):175–187, 2007.
21. K.-H. Yeap and M. Sarrafzadeh. Floor-planning by graph dualization: 2-concave rectilinear modules. *SIAM Journal on Computing*, 22:500–526, 1993.

Appendix

To prove correctness of the algorithm for proportional representations with 10-gons for maximal planar graphs, we need the following lemma:

Lemma 6. *Let G be a maximal plane graph and let Γ be the representation of G obtained by the algorithm of Section 2. Then for any two vertices u and v in G, the polygons representing u and v do not cross each other in Γ.*

Proof. Let v_1, v_2, v_3 be the three outer vertices of G and let T_1, T_2, T_3 be the three Schnyder trees rooted at v_1, v_2 and v_3, respectively. For each vertex v of G, Let $P(v)$ be the polygon representing v in Γ. By the choice of $\lambda(v)$, the bottommost bridge-strip is above the topmost foot-strip in Γ. Then by the construction, one can see that the only possible crossing might occur between a foot and a leg or between a foot and a body.

Let u and v be two vertices of G. We first assume that the foot of $P(u)$ crosses the leg of $P(v)$, as illustrated in Figure 6(a). Then u comes before v in the pre-order traversal of T_1; both u and v comes before both $f_2(u)$ and $f_2(v)$ in both the pre-order and the post-order traversals of T_1; and either $f_2(u) = f_2(v)$ or $f_2(u)$ comes before $f_2(v)$ in the post-order traversals of T_1. Let p_1 be the unique path from v to v_1, p_2 the leftmost path from v to one of its descendant leaf and p_3 the unique path from $f_2(v)$ to p_1 in T_1. Then u is to the right of the path $p_1 \cup p_2$ and $f_2(u)$ is inside the region enclosed by p_1, p_3 and the edge $(v, f_2(v))$, not on the path p_1. Then by the properties of Schnyder trees and by planarity, there cannot be any edge $(u, f_2(u))$, a contradiction.

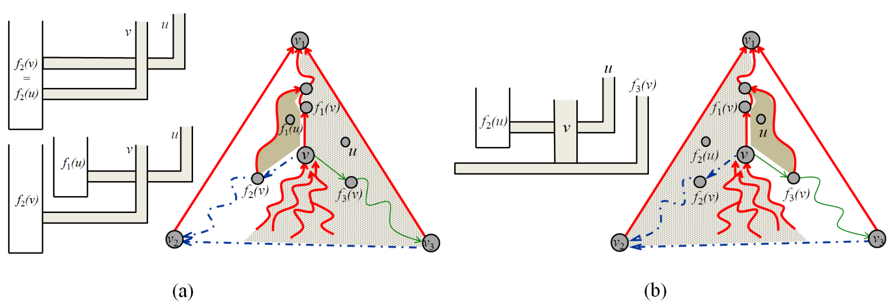

Fig. 6. Illustrations for the proof of Lemma 6.

We now assume that the foot of $P(u)$ crosses the body of $P(v)$. Then u precedes v in both the pre-order and the post-order traversals of T_1 and v precedes $f_2(u)$ in the post-order traversal of T_1, as illustrated in Figure 6(b). Had u been before $f_3(v)$ in the pre-order traversal of T_1, the the foot of u would cross the leg of $f_3(v)$, which is not possible according to the previous paragraph. We thus assume that u follows $f_3(v)$ in the pre-order traversal of T_1. Let p_1 be the unique path from v to v_1, p_2 the rightmost path from v to one of its descendant leaf and p_3 the unique path from $f_3(v)$ to p_1 in T_1. Then $f_2(u)$ is to the left of the path $p_1 \cup p_2$ and $f_2(u)$ is inside the region enclosed by p_1, p_3 and the edge $(v, f_3(v))$, not on the path p_1 and the edge $(v, f_3(v))$. Then by planarity, there cannot be any edge $(u, f_2(u))$, a contradiction. □

Illustration of the Algorithm for Maximal Planar Graphs

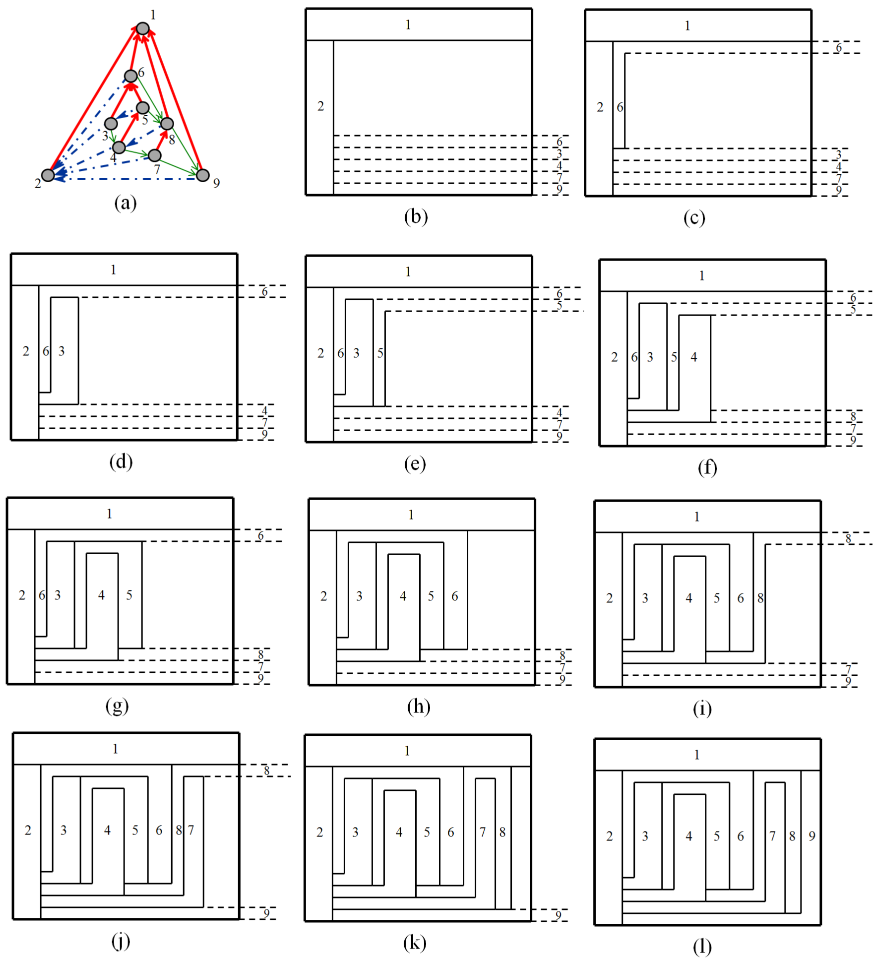

Fig. 7. Illustration of the algorithm for proportional contact representation of maximal planar graphs.

Illustration of the Algorithm for Planar 3-Trees

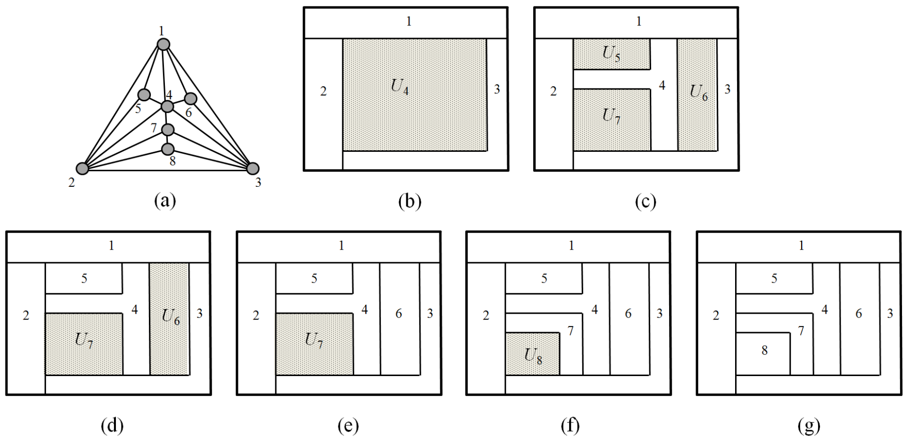

Fig. 8. Illustration of the algorithm for proportional contact representation of planar 3-trees.