Proportional Contact Representations of Planar Graphs
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Abstract. We study contact representations for planar graphs, with vertices represented by simple polygons and adjacencies represented by a point-contact or a side-contact between the corresponding polygons. Specifically, we consider proportional contact representations, where given vertex weights are represented by the areas of the corresponding polygons. Several natural optimization goals for such representations include minimizing the complexity of the polygons, the cartographic error, and the unused area. We describe optimal (with respect to complexity) constructive algorithms for proportional contact representations for general planar graphs and planar 2-segment graphs, which include maximal outer-planar graphs and partial 2-trees. Specifically, we show that: (a) 4-sided polygons are necessary and sufficient for a point-contact proportional representation for any planar graph; (b) triangles are necessary and sufficient for point-contact proportional representation of partial 2-trees; (c) trapezoids are necessary and sufficient for side-contact proportional representation of partial 2-trees; (d) convex quadrilaterals are necessary and sufficient for hole-free side-contact proportional representation for maximal outer-planar graphs.

1 Introduction

For both theoretical and practical reasons, there is a large body of work about representing planar graphs as *contact graphs*, i.e., graphs whose vertices are represented by geometrical objects with edges corresponding to two objects touching in some specified fashion. Typical classes of objects might be curves, line segments, or polygons. An early result is Koebe's theorem [20] that all planar graphs can be represented by touching disks.

In this paper, we consider contact graphs, with vertices represented by simple polygons and adjacencies represented by a point-contact or a side-contact between the corresponding polygons; see Fig. 1. In the weighted version of the problem, the input is not only a planar graph but also a weight function $w : V(G) \to R^+$ that assigns a weight to each vertex of $G = (V, E)$. A graph G admits a *proportional contact representation* with the weight function w if there exists a contact representation of G, where the area of the polygon for each vertex v of G is proportional to its weight $w(v)$. Such representations have practical applications in cartography, VLSI Layout, and floor-planning. Using adjacency of regions to represent edges in a graph can lead to a more compelling visualization than drawing a line segment between two points [3].

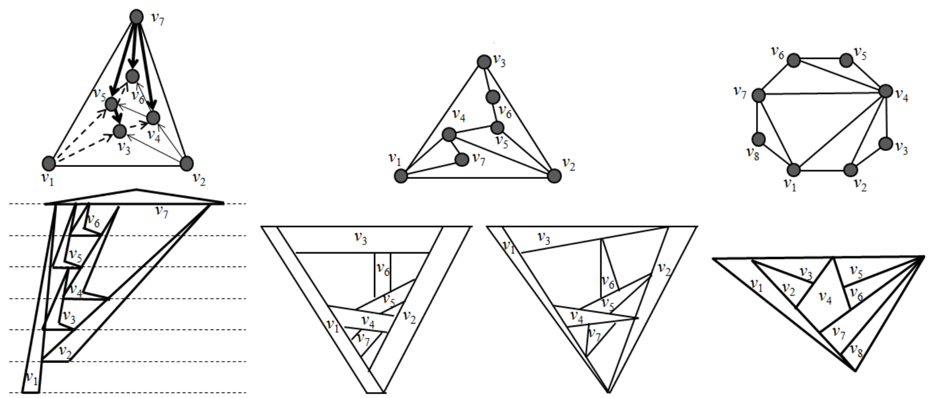

Fig. 1. (a) A planar graph and its proportional point-contact representation with four-sided non-convex polygons; (b) A 2-tree and its proportional side-contact representation with 4-sided convex polygons and proportional point-contact representation with triangles; (c) A maximal outer-planar graph and its hole-free proportional side-contact representation with 4-sided convex polygons.

In contact representations of planar graphs it is desirable, for aesthetic, practical and cognitive reasons, to limit how complicated the polygons are. In practical areas like VLSI layout, it is also desirable to minimize the unused area in the representation, also known as "holes". Another aspect of such representations is whether it uses convex polygons or not. With these considerations in mind, we study the problem of constructing proportional point-contact and side-contact representations of planar graphs w.r.t. the following parameters:

- *complexity*: maximum number of sides in a polygon representing a vertex;
- *cartographic error:* $\max_{v \in V} |A(v) - w(v)|$, where $A(v)$ is v's area, $w(v)$ its weight;
- *holes:* total unused area of the representation that is in the interior.

1.1 Related Work

Koebe's theorem [20] is an early example of point-contact representation and shows that a planar graph can be represented by touching circles. Any planar graph also has a contact representation where all the vertices are represented by triangles [5] and with cubes in 3D [9]. Kaufmann *et al.* [18] show that max-tolerance graphs have contact representations with homothetic triangles. Badent *et al.* [1] show that partial planar 3-trees and some series-parallel graphs also have contact representations with homothetic triangles. Recently, Gonçalves *et al.* [12] proved that any 3-connected planar graph and its dual can be simultaneously represented by touching triangles.

While the above results deal with point-contacts, the problem of constructing side-contact representations is less studied. Gansner *et al.* [10] show that any planar graph has a side-contact representation with convex hexagons. moreover, they show that 6 sides are necessary if convexity is required. The characterization of graphs admitting a hole-free side-contact representation with rectangles was obtained by Kozḿiński and Kinnen [21] or in the dual setting by Ungar [25]. Buchsbaum et al. [3] give an overview on the state of the art concerning rectangle contact graphs. It is also known that outer-planar graphs can be represented with side-contact representations of triangles [11].

Note that in all previous contact representation results, the areas of the circles or polygons is not considered. That is, these results deal with the unweighted version of the problem. Furthermore, previous work on contact representation does not consider whether holes are allowed, and if so, how big they will be. Our paper will take both area of shapes and existence of holes into account; we can show then that representations by triangles or even convex shapes are not possible.

Motivated by the application in VLSI layouts, contact representations of planar graphs with rectilinear polygons has also been studied and it is known that 8 sides are sometimes necessary and always sufficient (see e.g. [27]). Rahman *et al.* give an algorithm for hole-free proportional contact representation with 8-sided rectilinear polygons for a special class of plane graphs [23].

Another application of proportional contact representations can be found in *cartograms*, or value-by-area maps. Here, the goal is to redraw an existing geographic map so that a given weight function (e.g., population) is represented by the area of each country. Algorithms by van Kreveld and Speckman [26] and Heilmann *et al.* [16] yield representation with rectangular polygons and with zero or small cartographic errors but the adjacencies may be disturbed. De Berg *et al.* describe an algorithm for hole-free proportional contact representation with at most 40 sides for an internally triangulated plane graph G (and only 20 sides when G has four vertices on the exterior face and contains no separating triangles [4]. This was later improved to 34 sides [19] and then to 12 sides [2].

1.2 Our Results

In this paper we study the problem of proportional contact representation of planar graphs, while minimizing the complexity of the polygons, the cartographic error, and the unused area. The four main results in our paper are optimal (with respect to complexity) constructive algorithms for proportional contact representations for general planar graphs, outer-planar graphs, and partial 2-trees. Specifically, we show that: (a) 4-sided polygons are necessary and sufficient for a point-contact proportional representation for any planar graph; (b) triangles are necessary and sufficient for point-contact proportional representation of partial 2-trees; (c) trapezoids are necessary and sufficient for side-contact proportional representation of partial 2-trees; (d) quadrilaterals are necessary and sufficient for hole-free side-contact proportional representation for maximal outer-planar graphs.

Class of Graphs	Convexity	Complexity Lower Bound	Complexity Upper Bound	Hole-Free	Type of Contact
Planar	$\times$	4	4	$\times$	point
Planar 2-Trees	$\surd$	3	3	$\times$	point
Planar 2-Trees	$\surd$	4^*	4	$\times$	side
Maximal Outer-planar	$\surd$	4	4	$\surd$	side

Table 1. All results in this table are from in this paper, except the one marked $(^*)$, which follows from [11]. All algorithms are free of cartographic error.

2 Preliminaries

In a *point-contact representation* of a planar graph $G = (V, E)$, we construct a set P of closed simple interior-disjoint polygons with an isomorphism $\mathcal{P} : V \to P$ where for any two vertices $u, v \in V$, the boundaries of $\mathcal{P}(u)$ and $\mathcal{P}(v)$ touch at a *contact point* if and only if (u, v) is an edge. Let Γ be a contact representation of G. Then each interior face of G corresponds to a bounded hole in Γ and the exterior face of G corresponds to the unbounded hole in Γ. A *side-contact representation* of a planar graph is defined analogously, where instead of a contact point, we have a *contact side* between $\mathcal{P}(u)$ and $\mathcal{P}(v)$, which is a non-empty line segment in the boundary of both.

In a weighted version of the problem, the input also includes a weight function $w : V(G) \to R^+$ that assigns a positive weight to each vertex of G. We say that G admits a *proportional contact representation* with the weight function w if there is a contact representation of G where the area of the polygon for each vertex v of G is proportional to its weight $w(v)$. We define the *complexity of a polygonal region* as the number of sides it has.

A *plane graph* is a planar graph with a fixed embedding. A plane graph is *fully triangulated* or *maximally planar* if all its faces including the outerface are triangles. Both the concept of "canonical order" [6] and "Schnyder realizer" [24] are defined for fully triangulated plane graphs in the context of straight-line drawings of planar graphs on an integer grid. As we rely on these two concepts for our algorithms for proportional contact graph representations, we briefly review them below.

Let $G = (V, E)$ be a fully triangulated plane graph with outerface u, v, w in clockwise order. Then G has a *canonical order* of the vertices $v_1 = u$, $v_2 = v$, v_3, ..., $v_n = w$ which satisfies for every $4 \leq i \leq n$:

- The subgraph $G_{i-1} \subseteq G$ induced by v_1, v_2, ..., v_{i-1} is biconnected, and the boundary of its outer face is a cycle C_{i-1} containing the edge (u, v).
- The vertex v_i is in the exterior face of G_{i-1}, and its neighbors in G_{i-1} form an (at least 2-element) subinterval of the path $C_{i-1} - (u, v)$.

A *Schnyder realizer* of a fully triangulated graph G is a partition of the interior edges of G into three sets T_1, T_2 and T_3 of directed edges such that for each interior vertex v, the following conditions hold:

- v has out-degree exactly one in each of T_1, T_2 and T_3,
- the counterclockwise order of the edges incident to v is: entering T_1, leaving T_2, entering T_3, leaving T_1, entering T_2, leaving T_3.

The first condition implies that each T_i, $i = 1, 2, 3$ defines a tree rooted at exactly one exterior vertex and containing all the interior vertices such that the edges are directed towards the root. The following well-known lemma shows a profound connection between canonical orders and Schnyder realizers.

Lemma 1. *Let G be a fully triangulated plane graph. Then a canonical order of the vertices of G defines a Schnyder realizer of G, where the outgoing edges of a vertex v are to its first and last predecessor (where "first" is w.r.t. the clockwise order around v), and to its highest-numbered successor.*

3 Proportional Point-Contact Representations of Planar Graphs

In this section we show that 4-sided non-convex polygons are sometimes necessary and always sufficient for a proportional contact representation of a planar graph. We first describe an algorithm to obtain proportional point-contact representations of planar graphs using 4-sided non-convex polygons. We then show that there exists a planar graph with a given weight function that does not admit a proportional point-contact representation with convex polygons, let alone with 3-sided polygons, thus making our 4-sided construction optimal.

Theorem 1. *Let $G = (V, E)$ be a planar graph and let $w : V \to R^+$ be a weight function. Then G admits a proportional contact representation with respect to w in which each vertex of V is represented by a quadrilateral.*

Proof. We prove this claim constructively, showing how to generate a proportional contact representation of G with respect to w. We may assume that G is fully triangulated, for if it is not, we can add dummy vertices to make it so, and later remove those dummy vertices from the obtained proportional contact representation.

Assume after possible scaling that $w(v) \leq 1/n^2$ for all $v \in V$. We construct the drawing incrementally, following a canonical ordering $v_1, \ldots, v_n$. (We will often use j instead of v_j to simplify notation.) So that we don't have to change it later, we prescribe quite exactly what the quadrilateral assigned to j looks like before even placing it. Doing so will be easier using the notation of a Schnyder realizer, so let T_1, T_2, T_3 be the Schnyder realizer defined by the canonical ordering, with T_1 is rooted at 1, T_2 is rooted at 2 and T_3 is rooted at n. Let $\Phi_i(j)$ is the parent of j in tree T_i.

Assign an integer $\pi(j)$ to every vertex j such that $n \geq \pi(\phi_1(j)) > \pi(j) > \pi(\phi_2(j)) \geq 1$; this can be done since $T_2^{-1} \cup T_1$ is known to be acyclic, where T_2^{-1} is the tree T_2 with the direction of all its edges reversed. Now for every vertex $j \neq 1, 2, n$, we define the *spike* $\mathcal{S}(j)$ to be a quadrilateral with one reflex vertex. One segment (the *base*) is horizontal with y-coordinate j. Its length will be determined later, but it will always be at least $2w(j)$. From the left endpoint of the base, the spike continues with the *upward segment*, which has slope $\pi(j)$ and up to its *tip* which has y-coordinate $y = \Phi_3(j)$. Next comes the *downward segment* until the reflex vertex, and from there to the right endpoint of the base. See Figure 3(a). The placement of the reflex vertex is quite arbitrary, as long as the resulting shape has area $w(j)$ and the down-segment has positive slope. Note that since the base has length $\geq 2w(j)$ and y-coordinate j, the reflex vertex will have y-coordinate at most $j + 1$.

We first place $1, 2, n$, and then add $3, \ldots, n-1$ (in this order):

– Vertex 1 is represented by a triangle $\mathcal{S}(1)$ whose base has length $2w(1)/(n-1)$, placed arbitrarily with y-coordinate 1. The tip of $\mathcal{S}(1)$ has y-coordinate n.
– Vertex 2 is represented by a triangle $\mathcal{S}(2)$ whose base has length $2w(2)/(n-2)$, placed at y-coordinate 2 and with its left endpoint abutting $\mathcal{S}(1)$. The tip of $\mathcal{S}(2)$ has y-coordinate n.
– Vertex n is represented by a triangle whose base is at y-coordinate n and long enough to cover the tips of $\mathcal{S}(1)$ and $\mathcal{S}(2)$. We choose the height of $\mathcal{S}(n)$ such that the area is correct.

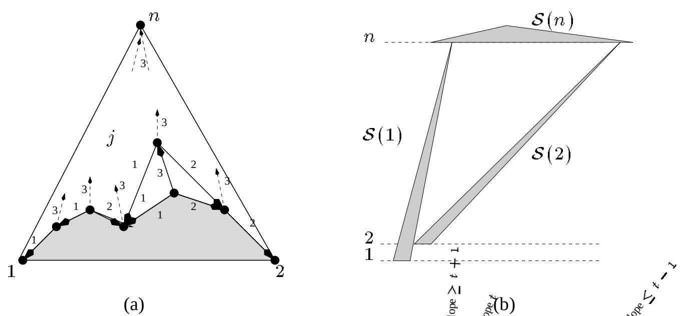

Fig. 2. (a) The canonical order and T_i (marked by labels); (b) the placement of 1,2,n.

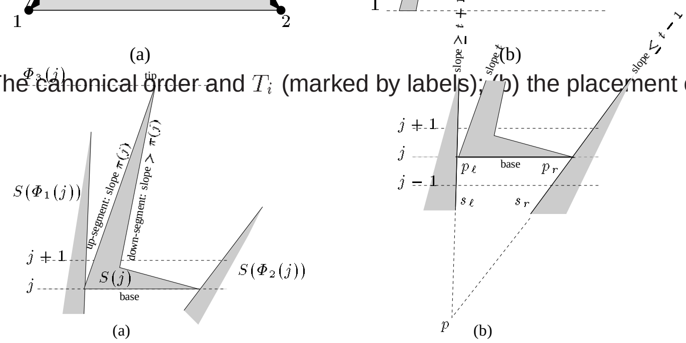

Fig. 3. (a) Adding j; (b) computing the width of the base.

Throughout placing vertices, we maintain the following invariant:

> For $j \geq 2$, after vertex j has been placed, the horizontal line with y-coordinate $j+1$ intersects only the spikes of the vertices on the outer-face of G_j, and in the order in which they are on the outer-face.

To place $j \geq 3$, we place the base of $\mathcal{S}(j)$ with y-coordinate j, and extend it from the down-segment of $\Phi_1(j)$ to the up-segment of $\Phi_2(j)$. Recall that $\Phi_2(j)$ and $\Phi_1(j)$ are exactly the first and last predecessor of j, and $j = \Phi_3(i)$ for all other predecessors $i \neq j$. Hence $\mathcal{S}(j)$ touches $\Phi_1(j)$ and $\Phi_2(j)$ at the ends of the base, and all other predecessors i of j have their tip at the base. So this creates a contact between j and all its predecessors. The rest of $\mathcal{S}(j)$ is then as described above.

This ends the description of the placement. It is straightforward to verify that the invariant holds, and therefore $\mathcal{S}(j)$ does not intersect any other spikes. To see that the base of $\mathcal{S}(j)$ is long enough, let p_ℓ and p_r be its left and right endpoint, and s_ℓ and s_r be the other segments containing them. Imagine that we extend s_ℓ and s_r until they meet in a point p. Since s_r contains a point with y-coordinate $\leq j-1$ (at the base of $\mathcal{S}(\Phi_2(j))$), triangle $\Delta\{p, p_\ell, p_r\}$ has height $h \geq 1$. See also Figure 3.

Let $t = \pi(v_j)$ be the slope of the up-segment of $\mathcal{S}(v_j)$. Since $\pi(\Phi_2(v_j)) < \pi(v_j) = t$, therefore s_r has slope at most $t-1$ and $x(p_r) \geq x(p) + \frac{h}{t-1}$. On the other hand, the slope of s_ℓ is positive by construction, and must exceed the slope of the up-segment of $\Phi_1(v_j)$, which has slope $\pi(\Phi_1(v_j)) > \pi(v_j) = t$. So s_ℓ has slope $\geq t+1$ and $x(p_i\ell) \leq x(p) + \frac{h}{t+1}$. Therefore,

$$x(p_r) - x(p_\ell) \geq \frac{h}{t-1} - \frac{h}{t+1} = \frac{h(t+1-(t-1))}{t^2-1} \geq \frac{2h}{t^2} \geq \frac{2}{n^2} \geq 2w(v_j)$$

where the last inequality holds since weights are small enough. Therefore the base of $\mathcal{S}(j)$ is wide enough, which ends the proof of the theorem. □

Our construction used non-convex shapes. We can show that this is sometimes required. The (somewhat technical) proof of the following lemma is in the appendix, and the graph for it is given in Figure 4 where the weight of the four small vertices is smaller than the weight of the four large vertices (a factor of 3 suffices).

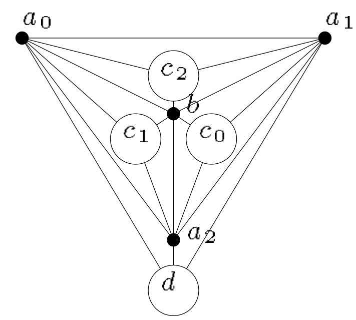

Fig. 4. Graph without prpoportional convex contact representation.

Lemma 2. *There exists a planar graph and a weight function such that the graph does not admit a proportional contact representation with respect to the weight function with convex shapes for all vertices.*

Lemma 2 also implies that using polygons of three sides are not always sufficient for proportional contact representations of planar graphs. On the other hand, Theorem 1 implies that any planar graph has a proportional contact representation with any given weight function on the vertices so that each of the vertices is represented by a non-convex polygon of at most four sides. Summarizing these two results we have the following theorem.

Theorem 2. *Four-sided non-convex polygons are always sufficient and sometimes necessary for proportional contact representation of a planar graph with a given weight function on the vertices.*

4 Subclasses of Planar Graphs with Convex-Shape Representations

In this section we address the problem of proportional contact representations for subclasses of planar graphs. The lower bound in Lemma 2 shows that for planar triangulations, the complexity in any proportional contact representation must be at least 4 and the polygons must be non-convex. We hence focus on planar graphs with fewer edges. In the next subsection we deal with proportional contact representations using triangles (or convex quadrilaterals for side-contacts.) Then we discuss hole-free representations for maximum outer-planar graphs.

4.1 2-Segment Graphs and Partial 2-Trees

Consider a planar graph that has an (unweighted) contact representation using line segments such that the intersection of any three segments is empty. We call this a *2-segment representation* and the graph a *2-segment graph*. We show that four-sided convex polygons are always sufficient and sometimes necessary for side-contact representations of these graphs. In fact, vertices will be represented by trapezoids. In case where only point-contacts are required, we show that 3 sides are sufficient (and, of course, necessary) for proportional contact representations of 2-segment graphs.

Theorem 3. *Let $G = (V, E)$ be a planar 2-segment graph. Then for any weight function $w : V \to R^+$ and any $\varepsilon > 0$, G has a proportional side-contact representation such that vertex v is represented by a trapezoid with area at least $w(v) - \varepsilon$ and at most $w(v)$.*

Proof. Let $\ell(v)$ be the line segment that represents v. We assume that ε is small enough such that "off-setting" any $\ell(v)$ by distance $\sqrt{\varepsilon}/2$ preserves adjacencies and does not create intersections. Here, *off-setting* $\ell(v)$ means moving it in parallel while shortening/lengthening it so that it still touches the segments at its ends. We also assume (after possible scaling) that $||\ell(v)|| \geq 2w(v)/\sqrt{\varepsilon} + \sqrt{\varepsilon}$ for all vertices v.

For any vertex v, create two copies of $\ell(v)$ that are off-set in parallel in both directions by so much that the trapezoid $T(v)$ between the two off-set lines has area $w(v)$. By the assumption on $||\ell(v)||$, this will require an off-set of les than $\sqrt{\varepsilon}/2$, hence adjacencies are preserved. So we get a proportional side-contact representation, except that $T(u)$ and $T(v)$ intersect for any edge (u, v).

To remove these intersections, let (u, v) be an edge, and assume that in the 2-segment representation, $\ell(u)$ ended at an interior point of $\ell(v)$. We then "retract" $T(u)$, i.e., we replace it by $T(u) - T(v)$. It remains to show that this does not disturb the area too much. Note that $T(u) \cap T(v)$ is a parallelogram, defined by $\ell(v)$ and one off-set line of $\ell(v)$, as well as the two off-set lines of $\ell(u)$. where the pairs of parallel lines have distance less than $\sqrt{\varepsilon}/2$ and $\sqrt{\varepsilon}$, respectively. Therefore, the area of $T(u) \cap T(v)$ is less than $\varepsilon/2$, and we remove such an area at each end of $T(u)$. This proves that the area of the retracted trapezoid is more than $w(u) - \varepsilon$ as desired. □

It is hence natural to ask for a characterization of 2-segment graphs. Thomassen gave one (Theorem 4) at Graph Drawing 1993 but never published his proof.

Theorem 4. *A planar graph $G = (V, E)$ is a 2-segment graph if and only if $|E[W]| \leq 2|W| - 3$ for every subset W of the vertices. As usual $E[W]$ denotes the set of edges with both ends in W.*

We provide a new proof of Theorem 4 based on rigidity theory in the appendix. The condition stated in the theorem can efficiently be checked (Lee and Streinu [22] provide a simple algorithm). In contrast, Hliněný [17] showed that the recognition of general contact graphs of segments is NP complete.

We call a graph G *2-shellable* if it is planar and 2-degenerate, i.e., has a vertex order $v_1, \ldots, v_n$ such that for $i \geq 3$ vertex v_i has at most two neighbours in $v_1, \ldots, v_{i-1}$. Such graphs have at most $2n - 3$ edges, hence by Theorem 4 a 2-shellable graph is a 2-segment graph. Moreover, from the proof it is easy to see that we may assume that the endpoints of segment $\ell(v)$ are adjacent to the predecessors of v for all vertices v. We can then create a proportional side-contact representation as above but without area-error by creating trapezoids in this vertex order. For each vertex v_i, first shorten $\ell(v_i)$ so that it ends at the off-set lines of v_i's predecessors. Then off-set $\ell(v_i)$ so that the resulting trapezoid has area $w(v_i)$. Some calculation shows that all off-sets are still at most $\sqrt{\varepsilon}/2$, so adjacencies are preserved.

Theorem 5. *Let $G = (V, E)$ be a 2-shellable graph and $w : V \rightarrow R^+$ be a weight function. Then G admits a proportional side-contact representation where each vertex of G is represented by a trapezoid with area $w(v)$.*

We derive two corollaries from Theorem 3 and 5. First, it is known that planar bipartite graphs are 2-segment graphs (we can even restrict the segments to be horizontal

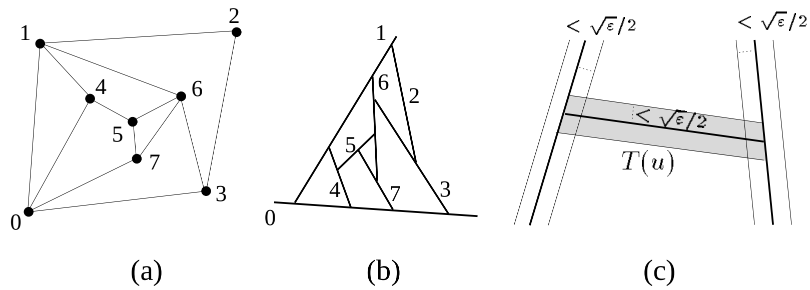

Fig. 5. (a&b) A 2-segment graph and representation; (c) converting to trapezoids.

or vertical) [15]. Hence they have a proportional side-contact representation with arbitrarily small carotgraphic error with trapezoids (in fact, rectangles.)

Second, a *2-tree* is either an edge or a graph G with a vertex v of degree two in G such that $G - v$ is a 2-tree and the neighbors of v are adjacent. A *partial 2-tree* is a subgraph of a 2-tree; this is the same as series-parallel graphs. Every partial 2-tree is planar. Directly from the definition we see that 2-trees (and hence partial 2-trees) are also 2-degerate, so they are 2-shellable. Therefore they have a proportional side-contact representation with trapezoids. We can also show that 4 sides are sometimes required.

Theorem 6. *Four-sided convex polygons are always sufficient and sometimes necessary for a proportional side-contact representation of a partial 2-tree with a given weight function.*

Proof. Sufficiency follows from Theorem 5, since partial 2-trees are 2-shellable. To establish necessary, consider the 2-tree obtained from $K_{2,4}$ by adding an edge between the vertices of the partition of size two. These two vertices then have four common neighbors. But as was proved in [11], in any side-contact representation with triangles, any pair of adjacent vertices has at most three neighbors. Hence our graph has no side-contact representation with triangles, let alone one that respects the weights. □

Note that if we move from side-contact representations to point-contact representations, we can reduce the complexity. Namely, replace line-segments by triangles such that only one endpoint of $\ell(v)$ gets moved (in both directions.) As in Theorem 3 we can hence prove (details are omitted):

Theorem 7. *Let $G = (V, E)$ be a 2-segment graph and $w : V \to R^+$ be a weight function. Then for any $\varepsilon > 0$, G admits a proportional point-contact representation where each vertex of G is represented by a triangle with area between $w(v)$ and $w(v) - \varepsilon$. If G is a 2-shellable graph, then the area of the triangle of v is exactly $w(v)$.*

4.2 Maximal Outer-planar Graphs

In this section, we study maximal outer-planar graphs, i.e., planar graphs whose outer-face is an n-cycle and all interior faces are triangles. These are 2-trees, so the results from the previous subsection apply, but (using a different construction) we can construct a side-contact represention using triangles that has no holes.

So assume that G is a maximal outer-planar graph. For any two vertices u, v denote by $G(u, v)$ the graph induced by the vertices that are between u to v (ends excluded) while walking along the outer-face in ccw order, and let $w(G(u,v))$ be the sum of the weights of all these vertices.

An *aligned triangle* is a triangle with horizontal base and tip below the base. This then naturally defines a *left* and *right side* of the triangle. The crucial idea is that we can represent an outer-planar graph inside *any* aligned triangle of suitable area.

Lemma 3. *Let $G = (V, E)$ be a maximal outer-planar graph and (u, v) an edge on the outer-face of G, with u before v in ccw order. Let $w : V \to R^+$ be a weight-function. Then for any aligned triangle T of area $w(G(v, u))$, there exists a hole-free proportional side-contact representation of $G(v, u)$ inside T such that the left [right] side of G contains segments of the neighbors of u [v] and of no other vertices.*

Proof. We proceed by induction on the number of vertices in G. In the base case, G is a 3-cycle $\{u, v, x\}$. Use T as shape for x; this satisfies all conditions.

In the step, let x be the unique common neighbour of u and v. Divide T with a segment s from the tip to the base such that the region T_ℓ left of s has area $w(G(x, u)) + \frac{1}{2}w(x)$, and the region T_r right of ℓ has area $w(G(v, x)) + \frac{1}{2}w(x)$. Cut off triangles of area $\frac{1}{2}w(x)$ each from the tips of T_ℓ and T_r; the combination of these two triangles forms a convex quadrilateral of area $w(x)$ which we use for x. See Figure 6. Recursively place $G(x, u)$ and $G(v, x)$ (if non-empty) in the triangles of T that remain; one easily verifies that these have the correct area and that we obtained the desired side-contact representation. □

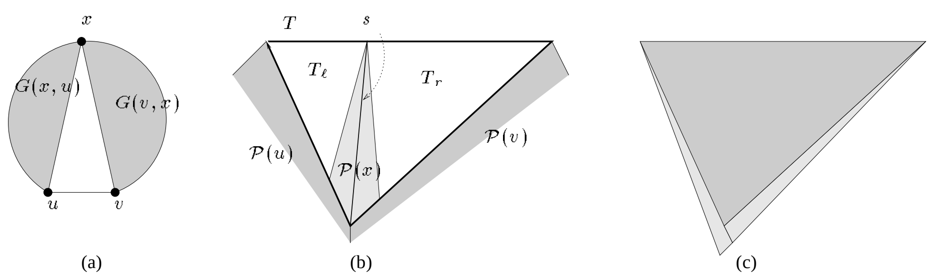

Fig. 6. The construction for maximal outer-planar graphs: (a) the graph; (b) splitting triangle T suitably; (c) adding u and v in the outer-most recursion.

Applying this lemma for an arbitrary edge (u, v) on the outer-face and arbitrary triangle T of appropriate area then gives a drawing of $G(v, u)$; we can add triangles for u and v to it to complete it to a contact representation of G, and hence obtain the result.

Corollary 1. *Let $G = (V, E)$ be a maximal outer-planar graph and let $w : V \to R^+$ be a weight function. Then G admits a hole-free proportional side-contact representation where vertices are represented by convex quadrilaterals.*

We now show that the representation obtained by this algorithm is also optimal for a maximal outerplanar graph with respect to complexity. To do this we used the *snowflake graph* S, which is the graph obtained from a triangle by repeatedly walking around the outer-face and adding a vertex of degree 2 at each edge; see Fig. 7(a).

Lemma 4. *S has no hole-free side-contact representation with triangles that all have the same area.*

Sketch of Proof. Assume for contradiction that there is such a representation Γ. Let S_i be the vertices added when we walk around the outer-face for the ith time; we call this the *ith level*. One can observe that all the angles in the outer-boundary of the Γ are concave except for at most four convex corners. Since the number of vertices doubles on each level, there must be two triangles T and T' on adjacent levels such that the base of T (by which we mean the side that was exposed after adding T) is at least twice the length of the base of T'. Since both triangles have equal area, a simple calculation involving adjacent angles shows that this is a contradiction; see Fig. 7(b). □

By Corollary 1 and Lemma 4, we have the following theorem.

Theorem 8. *Convex quadrilaterals are always sufficient and sometimes necessary for hole-free proportional side-contact representations of maximal outer-planar graphs.*

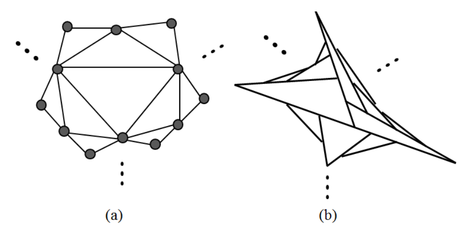

Fig. 7. (a) The snowflake graph S; (b) illustration for the proof of Lemma 4.

5 Conclusion and Open Problems

We described several constructive algorithm for proportional point-contact and side-contact representations of planar graphs, outer-planar graphs, and 2-trees. We focused on the complexity of the polygons representing vertices, and provide bounds on this complexity that are tight, for a variety of graph classes and drawing models.

However, many problems still remain open. Most interesting is: what is the complexity of side-contact proportional representations of maximal planar graphs? We can achieve 8-sided polygons easily (essentially by cutting the ends of the 4-sided spikes), but can we do less? Likewise, what is the complexity for hole-free proportional representations of maximal planar graphs? Here, a bound of 12 is known (and the polygons are orthogonal) [2], but can we do better if polygons need not be orthogonal (or perhaps even if they do)? Vice versa, can we show a lower bound on the complexity?

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Appendix

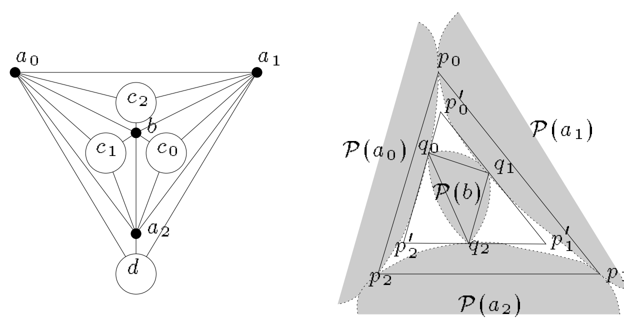
Proof of Lemma 2. We aim to show that the graph in Figure 8(a) has no proportional representation with convex polygons if the small vertices have weight δ and the larger vertices have weight $D > 3\delta$. Assume for contradiction that we had such a representation. Note that this graph is 3-connected and all faces of this graph are isomorphic (even when taking vertex weights into account), so all planar embeddings of it are equivalent. We may assume therefore that d is in the outer-face. We will focus now on the sub-graph defined by a_0, a_1, a_2 and its interior. See Figure 8(b) for an illustration of the following notation.

For $i = 0, 1, 2$, let p_i be a point of contact between $\mathcal{P}(a_i)$ and $\mathcal{P}(a_{i+1})$ (where addition is modulo 3.) Further, let q_i be a point of contact between $\mathcal{P}(a_i)$ and $\mathcal{P}(b)$. Define T_0 to be the triangle $\Delta\{p_0, p_1, p_2\}$ and T_2 to be the triangle $\Delta\{q_0, q_1, q_2\}$. Let T_1 be the triangle obtained by moving the edges of T_0 parallel inward until the resulting triangle circumscribes T_2, i.e., until its sides contain q_0, q_1 and q_2. Let p'_i be the corner of T_1 that corresponds to the corner p_i of T_0.

Now we analyze the areas of various triangles defined by these points. First, triangle $\Delta\{p_0, p_1, q_1\} \subseteq \mathcal{P}(a_1)$ by convexity, so it has area at most δ. Next, triangle $\Delta\{p_1, p'_1, q_1\}$ has the same height and a not-larger base than $\Delta\{p_0, p_1, q_1\}$, so the area of $\Delta\{p_1, p'_1, q_1\}$ is at most δ. Similarly one shows that triangle $\Delta\{p_1, q_2, p'_1\}$ has area at most δ.

Now consider triangle $\Delta\{p_1, q_2, q_1\}$; this contains $\mathcal{P}(c_0)$ and hence has area at least D. Therefore triangle $\Delta\{p'_1, q_2, q_1\} = \Delta\{p_1, q_2, q_1\} - \Delta\{p_1, q_2, p'_1\} - \Delta\{p_1, p'_1, q_1\}$ has area at least $D - \delta - \delta > \delta$ (by choice of $D \geq 3\delta$.) Similarly one shows that triangle $\Delta\{p'_2, q_0, q_2\}$ and triangle $\Delta\{p'_0, q_1, q_0\}$ have area strictly greater than δ.

Finally, observe that $T_0 \subseteq \mathcal{P}(b)$, and hence T_0 has area at most δ. But now we have a triangle T_0 of area at most δ that is circumscribed by a triangle T_1 such that the three triangles of $T_1 - T_0$ each have area strictly greater than δ. This is impossible by a very old result from geometry; see e.g. [7]. □



(a) (b)

Fig. 8. A graph G, and proving that it does not have a proportional contact-representation with convex shapes.

Proof of Theorem 4. We aim to show a planar graph is a 2-segment graph if and only if for any $W \subseteq V$, we have $|E[W]| \leq 2|W| - 3$. The necessity of the condition is easily seen. Let $\mathcal{S}$ be the set of segments of a 2-segment representation of G. For $W \subset V$ let X_W be the set of end-points of segments in $\mathcal{S}$ corresponding to vertices of W. Since we have a 2-segment representation we may assume that $|X_W| = 2|W|$. There is an injection ϕ from edges in $E[W]$ to points in X_W, points belonging to the convex hull of X_W, however, can not be in the image of ϕ. Since the convex hull contains at least three points we get: $|E[W]| \leq |X_W| - 3 = 2|W| - 3$.

For the converse we need some prerequisites. A Laman graph is a graph $G = (V, E)$ with $|E| = |V| - 3$ and $|E[W]| \leq |W| - 3$ for all $W \subset V$. Laman graphs are of interest in rigidity-theory, see e.g. [13, 8]. Laman graphs admit a planar Henneberg construction, i.e., an ordering $v_1, \ldots, v_n$ of the vertices such that if G_i is the graph induced by the vertices $v_1, \ldots, v_i$ then G_3 is a triangle and G_i is obtained from G_{i-1} by one of the following two operations:

($\mathbf{H_1}$) Choose two vertices x, y from G_{i-1} and add v_i together with the edges (v_i, x) and (v_i, y).

($\mathbf{H_2}$) Choose an edge (x, y) and a third vertex z from G_{i-1}, remove (x, y) and add v_i together with the three edges (v_i, x), (v_i, y), and $(v_i, z$.

In [14] it is shown that planar Laman graphs admit a planar Henneberg construction in the sense that the graph is constructed together with a plane staight-line embedding and vertices stay at their position once they have been inserted.

Now let G be a planar graph fulfilling the condition of the theorem. We may assume that G is Laman since we can easily get rid of edges in a segment contact representation by retracting ends of segments. Consider a planar Henneberg construction $G_3, \ldots G_n$. Starting from three pairwise touching segments representing G_3 we add segments one by one. For the induction we need the invariant that after adding the ith segment s_i we have a 2-segment representation of G_i and there is a correspondence between the cells of the segment representation and the faces of G_i which preserves edges, i.e., if (x, y) is an edge of the face, then one of the corners of the corresponding cell is a contact between s_x and s_y. Figure 9 indicates how to add segment s_i in the cases where v_i is added by $\mathbf{H_1}$, resp. $\mathbf{H_2}$. It is evident that the invariant for the induction is maintained. □

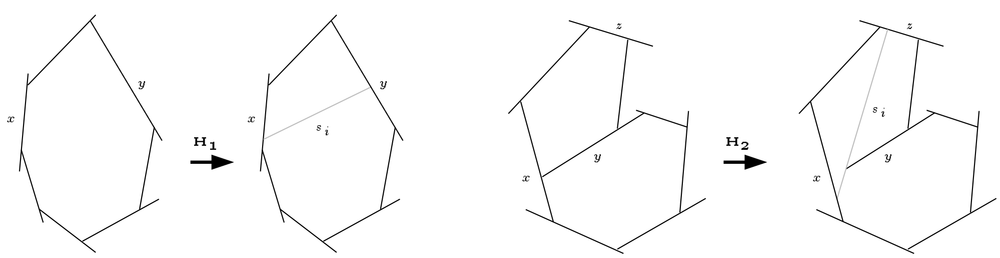


Fig. 9. The addition of segment s_i.