Drawing planar 3-trees with given face-area* (Technical Report CS-2009-20)

Therese Biedl[1] and Lesvia Elena Ruiz Velázquez[1]

David R. Cheriton School of Computer Science, University of Waterloo, Waterloo, ON N2L 3G1, Canada, {biedl,leruizve}@uwaterloo.ca.

Abstract. We study straight-line drawings of planar graphs such that each interior face has a prescribed area. It was known that such drawings exist for all planar graphs with maximum degree 3. We show here that such drawings exist for all planar partial 3-trees. Moreover, vertices have rational coordinates if the face-areas are rational, and we can bound the resolution. We also give some negative results for other graph classes.

1 Introduction

A planar graph is a graph that can be drawn without crossing. Fáry, Stein and Wagner [4, 9, 12] proved independently that every planar graph has a drawing such that all edges are drawn as straight-line segments. Sometimes additional constraints are imposed on the drawings. The most famous one is to have integer coordinates while keeping the area small; it was shown in 1990 that this is always possible in $O(n^2)$ area [5, 8]. Another restriction might be to ask whether all edge lengths are integral; this exists if the graph is 3-regular [6], but is open in general.

In this paper, we consider drawings with prescribed face areas. This has applications in cartograms, where faces (i.e., countries in a map) should be proportional to some property of the country, such as population. Ringel [7] showed that such drawings do not exist for all planar graphs. Thomassen [10] showed that they do exist for planar graphs with maximum degree 3. Quite a few results are known for drawings with prescribed face areas that are not straight-line, but instead use orthogonal paths, preferably with few bends [11, 1, 3].

We show that every planar partial 3-tree, for any given set of face areas, admits a planar straight-line drawing that respects the face areas. Our main contribution is that such drawings not only exist, but that the coordinates are rational (presuming the face-areas are,) and that we can bound the least common denominator (albeit not polynomially.)

It remains open whether Thomassen's proof could be modified to yield rational coordinates for all planar graphs of maximum degree 3; we provide some evidence why this seems unlikely. We also show that planar partial 4-trees sometimes cannot be realized at all, and sometimes only with irrational coordinates.

2 Background

Let $G = (V, E)$ be a graph with n vertices and m edges that is *simple* (has no loops or multiple edges) and *planar* (can be drawn without crossing.) A planar

* Research supported by NSERC.

drawing of G splits the plane into connected pieces; the unbounded piece is called the *outer-face*, all other pieces are called *interior faces.* We assume that one combinatorial drawing (characterized by the clockwise order of edges around each vertex and choice of the outer-face) has been fixed for G.

A *planar straight-line drawing* of G is an assignment of vertices to distinct points in the plane such that no two (induced) straight-line segments of edges cross, and the fixed order of edges and outer-face are respected.

Let A be a function that assigns non-negative rationals[1] to interior faces of G. We say that a planar straight-line drawing of G *respects the given face areas* if every interior face f of G is drawn with area $const \cdot A(f)$, where the constant is the same for all faces. If $A \equiv 1$, then the drawing is called an *equifacial drawing.*

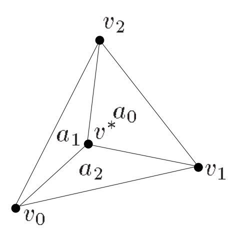
A graph G is a *k-tree* if it has a vertex order $v_1, \ldots, v_n$ such that for $i > k$ vertex v_i has exactly k earlier neighbours, and they form a clique. A *partial k-tree* is a subgraph of a k-tree. Assume G is a planar 3-tree. Then vertex v_i (for $i > 3$) has three *predecessors*, i.e., earlier neighbours, and they form a triangle. Hence we can think of G as being built up by starting with a triangle, and repeatedly picking a face f (which is necessarily a triangle) and subdividing f into three triangles by inserting a new vertex in it. One can show that the first triangle in this process can be presumed to be the outer-face.

A planar partial 3-tree is a graph G' that is planar and is the subgraph of a 3-tree G. It is not obvious that G can be assumed to be planar (for example, this is not true if we replace '3' by '4'), but one can show that this is indeed true (all crucial ingredients for this are in [2].) Since "drawing with prescribed face areas" is a property that is closed under taking subgraphs (see also Lemma 3), we hence mostly focus on drawing planar 3-trees.

3 Drawing planar partial 3-trees

We now show that every planar partial 3-tree can be drawn with given face areas. A vital ingredient is how to draw K_4 by placing one point inside a triangle.

Lemma 1. *Let T be a triangle with area a and vertices v_0, v_1, v_2 in counterclockwise order. For any non-negative value $a_0+a_1+a_2 = a$, there exists a point v^* inside T such that triangle $\{v_{i+1}, v_{i-1}, v^*\}$ has area a_i, for $i = 0, 1, 2$ and addition modulo 3.*



Proof. Let (x_0, y_0), $(x_1, y_1), (x_2, y_2), (x^*, y^*)$ be the coordinates of v_0, v_1, v_2, v^*, respectively. The signed area formula expresses the area of a triangle via determinants; the result is positive if the vertices are counterclockwise around the triangle and negative otherwise. In particular, for a_i to be the area of a triangle $\{v_{i+1}, v_{i-1}, v^*\}$ (for $i = 0, 1, 2$ and addition modulo 3), we must have

[1] Irrational face areas could be allowed, but would force irrational coordinates.

$$2 \cdot a_i = \begin{vmatrix} x_{i+1} & y_{i+1} & 1 \\ x_{i-1} & y_{i-1} & 1 \\ x^* & y^* & 1 \end{vmatrix}$$
$$= (x_{i-1} \cdot y^* - x^* \cdot y_{i-1}) - (x_{i-1} \cdot y_{i+1} - x_{i+1} \cdot y_{i-1}) + (x^* \cdot y_{i+1} - x_{i+1} \cdot y^*)$$

Since the triangle defined by v_0, v_1, v_2 has area $a = a_1 + a_2 + a_3$, we also know

$$2 \cdot a = \begin{vmatrix} x_0 & y_0 & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = (x_1 \cdot y_2 - x_2 \cdot y_1) - (x_1 \cdot y_0 - x_0 \cdot y_1) + (x_2 \cdot y_0 - x_0 \cdot y_2)$$

Combining these equations yields after sufficient manipulation that

$$x^* = \frac{a_1 \cdot x_1 + a_2 \cdot x_2 + a_3 \cdot x_3}{a_1 + a_2 + a_3} \quad \text{and} \quad y^* = \frac{a_1 \cdot y_1 + a_2 \cdot y_2 + a_3 \cdot y_3}{a_1 + a_2 + a_3} \tag{1}$$

Since $2a_i$ is non-negative, the signed-area formula guarantees that v^* lies to the left of the directed segments v_0v_1, v_1v_2, and v_2v_0, and hence inside T. □

Lemma 2. *Every planar 3-tree can be drawn respecting prescribed face areas.*

Proof. Assume $v_1, \ldots, v_n$ is the vertex-order that defined the 3-tree G, with $\{v_1, v_2, v_3\}$ the outer-face. We proceed by induction on n. The base case is $n = 3$, where this is obvious. If $n \geq 4$, then consider the K_4 formed by v_n and its neighbours. In $G - v_n$, these neighbours form a triangle T that is an interior face. Draw $G - v_n$ recursively, requiring as area for T the sum of the area of the faces around v_n. Then, by Lemma 1, v_n can be added inside T suitably. □

Lemma 3. *Every planar partial 3-tree can be drawn respecting prescribed face areas.*

Proof. Add edges to convert the graph into a planar 3-tree G. Each time an edge is added, it divides a face f_i into two faces f_i^1 and f_i^2. Let a_i be the prescribed area for f_i, then we choose area a_i^j for face f_i^j such that $a_i^1 + a_i^2 = a_i$, e.g. $a_i^1 = a_i^2 = \frac{a_i}{2}$. By Lemma 2, G can be drawn respecting the prescribed face areas. Deleting all added edges then gives the desired drawing. □

In our construction, we are interested not only in whether such a drawing exists, but what bounds can be imposed on the resulting coordinates. (This was not studied at all in the previous literature.) If all areas are rationals, then Equation (1) shows immediately that the newly placed vertex v^* has rational coordinates if the coordinates of T are rational. Hence, using induction and starting in the base case with a triangle with rational coordinates, one can immediately show that all coordinates of all vertices are rational. We summarize:

Theorem 1. *Let G be a planar partial 3-tree and A be an assignment of non-negative rationals to interior faces of G. Then G has a straight-line drawing such that each interior face f of G has area $A(f)$ and all coordinates are rationals.*

We can also give bounds on the required resolution.

Theorem 2. *Any planar 3-tree G has an equifacial straight-line drawing with integer coordinates and width and height at most $\prod_{k=1}^{n}(2k+1)$.*

Proof. We show that G has an equifacial straight-line drawing with rational coordinates in $[0,1]$ with common denominator at most $\prod_{k=1}^{n}(2k+1)$; the result then follows after scaling. Let $v_1,\dots,v_n$ be a vertex order of G with v_1,v_2,v_3 the outer-face. The drawing is the one from Theorem 1; we assume that v_1,v_2,v_3 are at the triangle $T=\{(1,0),(0,1),(0,0)\}$ (this can be enforced in the base case of Lemma 2.) Since G is triangulated, it has $2n-5$ faces; so each interior face is drawn with area $a=1/(4n-10)$ since T has area $1/2$. We show the bound on the denominator only for x-coordinates; y-coordinates are proved similarly.

We need some notations. Recall that we can view graph G as being obtained by inserting vertex v_j into the triangle T_j spanned by the three predecessors of v_j. Let G_j be the subgraph of G induced by all vertices on or inside T_j. Since T_j was a face in the graph induced by $\{v_1,\dots,v_{j-1}\}$, all vertices in G_j are either v_j, or one of its three predecessor, or a vertex in $\{v_{j+1},\dots,v_n\}$ and so G_j has at most $n-j+4$ vertices. Let f_j be the number of interior faces in G_j; we have $f_j\le 2(n-j+4)-5=2n-2j+3$. Also note that T_j contains exactly these f_j faces and they all have area $1/(4n-10)$, so T_j has area $f_j/(4n-10)$.

We will show by induction on i that vertex v_i has x-coordinate

$$x_i=\frac{integer}{\prod_{4\le j\le i} f_j} \tag{2}$$

for some integer that we will not analyze further to keep notation simple. Nothing is to show for $i=1,2,3$, since x_i is an integer by choice of the points for the outer-face triangle. For $i\ge 4$, let v_{i_0},v_{i_1},v_{i_2} be the three predecessors of v_i.

For $k=0,1,2$, Equation (2) holds for x_{i_k} by $i_k\le i-1$ and induction, and expanding with integers $f_{i_k+1},\dots,f_{i-1}$ yields

$$x_{i_k}=\frac{integer}{\prod_{4\le j\le i_k} f_j}=\frac{integer}{\prod_{4\le j\le i-1} f_j}$$

Equation (1) states that $x_i=(a_0x_{i_0}+a_1x_{i_1}+a_2x_{i_2})/(a_0+a_1+a_2)$, where a_0,a_1,a_2 are the areas of faces incident to v_i. For $k=0,1,2$, each a_k is the sum of faces in some subgraph, and therefore an integer multiple of $1/(4n-10)$. Furthermore, $a_0+a_1+a_2$ is exactly the area of triangle T_i spanned by v_{i_1},v_{i_2},v_{i_3}, which we argued earlier is $f_i/(4n-10)$. Hence, as desired,

$$x_i=\frac{a_0x_{i_0}+a_1x_{i_1}+a_2x_{i_2}}{a_0+a_1+a_2}=\frac{\sum_{k=0}^{2}\frac{integer}{4n-10}\frac{integer}{\prod_{4\le j\le i-1} f_j}}{\frac{f_i}{4n-10}}=\frac{integer}{\prod_{4\le j\le i} f_j}.$$

Since $f_4,\dots,f_n$ are integers, by Equation (2) all x_i's have common denominator

$$\prod_{4\le j\le n} f_j \le \prod_{4\le j\le n}(2n-2j+3)=\prod_{k=1}^{n-3}(2k+1) \qquad \square$$

Two remarks. First, we can obtain similar (but uglier-looking) bounds for arbitrary integer face areas, by replacing 'f_j' by 'the sum of the f_j largest face areas in G'. Second, we did experiments to see whether our bounds are tight. We computed (using Maple) the coordinates in Theorem 2 for the planar planar 3-tree $v_1, \ldots, v_n$ where v_i has predecessors $v_{i-1}, v_{i-2}, v_{i-3}$ for $i \geq 4$; note that this graph has $f_i = 2n - 2i + 3$ and hence is a good candidate to obtain the bound in Theorem 2. Figure 1 shows the least common denominator for various values of n; they are smaller than the upper bound but are clearly growing in exponential fashion as well.

n	LCD in drawing	upper bound
10	$5.0 \cdot 10^{3}$	$2.0 \cdot 10^{6}$
50	$3.1 \cdot 10^{34}$	$2.8 \cdot 10^{75}$
100	$1.0 \cdot 10^{82}$	$1.7 \cdot 10^{183}$
500	$1.0 \cdot 10^{427}$	$2.0 \cdot 10^{1271}$
1000	$2.8 \cdot 10^{852}$	$4.8 \cdot 10^{2853}$

Fig. 1. Lower and upper bounds on the resolution in the drawing.

4 Negative results

In this section, we give some examples of graphs where no realization with rational coordinates is possible, hence providing counter-example to some possible conjectured generalizations of Theorem 1.

The first example is the octahedron where all face areas are 1 except for two non-adjacent, non-opposite faces, which have area 3. As shown by Ringel [7], any drawing that respects these areas must have some complex coordinates. (Ringel's result was actually for the graph G_1 obtained from the octahedron by subdividing two triangles further; the resulting graph then has no equifacial drawing.) Note that both the octahedron and G_1 are planar partial 4-trees, so not all partial 4-trees have equifacial drawings.

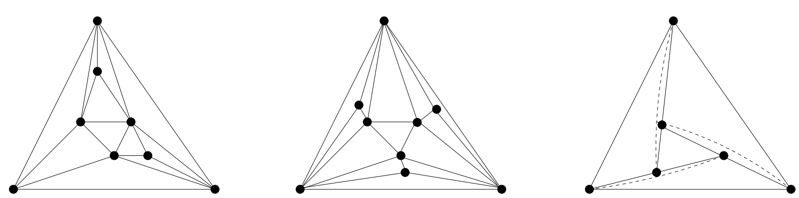


Fig. 2. Graphs G_1, G_2 and G_3.

The second example is the octahedron where all face areas are 1 except that the three faces adjacent to the outer-face have area 3. (Alternatively, one could ask for an equifacial drawing of graph G_2 in Figure 2.) Assume, after possible linear transformation, that the vertices in the outer-face are at $(0, 0)$, $(0, 13)$ and

$(2, 0)$. Computing the signed area of all the faces one can show that the vertices not on the outerface are at $(\frac{10}{3} + \frac{2\sqrt{3}}{13}, 5 - \sqrt{3}), (\frac{10}{3} - \frac{2\sqrt{3}}{13}, 3)$ and $(\frac{6}{13}, 5 + \sqrt{3})$. Thus even if a partial 4-tree has an equifacial drawing, it may not have one with rational coordinates.

The third example is again the octahedron, with three face areas prescribed to be 0, which forces some edges to be aligned as shown in Figure 2. If all other interior faces have area 1/8, and the outer-face is at $(1, 0), (0, 1), (0, 0)$, then similar computations show that some of the coordinates of the other three vertices are $(3 \pm \sqrt{5})/8$. Let G_3 be the graph obtained from the octahedron by deleting the edges that are dashed in Figure 2. Graph G_3 is a crucial ingredient in Thomassen's proof [10] that every planar of maximum degree 3 graph has a straight-line drawing with given face areas: in one case he splits the input graph into G_3 and three subgraphs inside three interior faces of G_3, draws G_3 with the edges aligned as in Figure 2, and recursively draws and pastes the subgraphs. Since we showed that G_3 cannot always be drawn with rational coordinates, then Thomassen's proof, as is, does not give rational coordinates. It remains an open problem whether Thomassen's proof could be modified to show that any planar graph with maximum degree 3 has a drawing respecting given rational face-areas that has rational coordinates.

References

1. Mark de Berg, Elena Mumford, and Bettina Speckmann. On rectilinear duals for vertex-weighted plane graphs. In *Graph Drawing (GD'05)*, volume 3843 of *Lecture Notes in Computer Science*, pages 61–72, 2006.
2. Ehab S. El-Mallah and Charles J. Colbourn. On two dual classes of planar graphs. *Discrete Mathematics*, 80(1):21–40, 1990.
3. D. Eppstein, E. Mumford, and B. Speckmann. Area-universal rectangular layouts. In *Symposium on Computational Geometry*, 2009. To appear.
4. I. Fáry. On straight line representation of planar graphs. *Acta. Sci. Math. Szeged*, 11:229–233, 1948.
5. H. de Fraysseix, J. Pach, and R. Pollack. How to draw a planar graph on a grid. *Combinatorica*, 10:41–51, 1990.
6. J. Geelen, A. Guo, and D. McKinnon. Straight line embeddings of cubic planar graphs with integer edge lengths. *Journal of Graph Theory*, 58(3):270–274, 2008.
7. G. Ringel. Equiareal graphs. In *Contemporary methods in graph theory. In honour of Prof. Dr. K. Wagner*, pages 503–505. Wissenschaftsverlag, Mannheim, 1990.
8. W. Schnyder. Embedding planar graphs on the grid. In *1st Annual ACM-SIAM Symposium on Discrete Algorithms*, pages 138–148, 1990.
9. S. Stein. Convex maps. In *Amer. Math. Soc.*, volume 2, pages 464–466, 1951.
10. Carsten Thomassen. Plane cubic graphs with prescribed face areas. *Combinatorics, Probability & Computing*, 1:371–381, 1992.
11. Marc J. van Kreveld and Bettina Speckmann. On rectangular cartograms. In *ESA*, volume 3221 of *Lecture Notes in Computer Science*, pages 724–735. Springer, 2004.
12. K. Wagner. Bemerkungen zum Vierfarbenproblem. *Jahresbericht der Deutschen Mathematiker-Vereinigung*, 46:26–32, 1936.