Edge-intersection graphs of k-bend paths in grids

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Abstract. In this paper, we continue the study of edge-intersection graphs of paths in a grid, which was initiated by Golumbic, Lipshteyn and Stern. We show that for any k, if the number of bends in each path is restricted to be at most k, then not all graphs can be represented. Then we study some graph classes that can be represented with k-bend paths, for small values of k.

1 Introduction

Presume you have a network and need to route calls in it. Since calls interfere with each other, you need to route calls such that no two connections share a link. This transforms into a colouring problem in an edge-intersection graph of paths. The network is the host graph, and each call becomes a path in the host graph; calls share a link if and only if the paths share an edge of the host graph. The edge-intersection graph defined by this has a vertex for every path, and vertices are adjacent if and only if the paths share an edge; the goal is hence to colour this edge-intersection graph.

Edge-intersection graphs of paths in a network have been studied almost exclusively for networks that are trees; this graph class is known as *EPT graphs* and was introduced over 20 years ago, though research is still ongoing (see [5] and the references therein.)

Very recently, Golumbic, Lipshteyn and Stern generalized the framework by allowing networks that are grids, rather than trees [6]. Thus, they define *EPG graphs* to be the edge-intersection graphs of paths in a grid. They showed that every graph is an EPG graph, and then restricted the graph class by limiting the number of *bends* for the paths, i.e., the number of times that the direction of the path switches from horizontal to vertical. Focusing on the case of single bends, they proved a number of existence and impossibility results.

In this paper, we continue this work, and study graphs that can be represented with few (but more than 1) bends in each path. We first show that for any given k, there exist graphs that are not *k-bend EPG graphs.* We then study some special graph classes and show that these are k-bend EPG graphs for small values of k.

2 Definitions

We assume familiarity with graph theory notation, see for example Golumbic's book [4]. In this paper, *grid* is used to denote the 2-way infinite orthogonal grid, i.e., the vertices are all points in 2D with integer coordinates, and grid-points are adjacent if they have distance 1. A *(grid) path* is a path in the grid. A *bend* is a place where a grid path changes direction. A *(grid) segment* is a grid path without any bends. We say that two grid paths *intersect* if they share at least one edge, otherwise they are *disjoint.* (Note that disjoint grid paths may still share a vertex of the grid, but not an edge.) We will omit "grid" whenever this does not lead to confusion.

An *EPG representation* of a graph $G = (V, E)$ is an assignment of grid paths to vertices of G such that (v, w) is an edge if and only if the paths assigned to v and w share a grid edge. We call a representation a *k-bend EPG representation* if every grid path representing a vertex has at most k bends. A graph is called a *k-bend EPG* graph if it has a k-bend EPG representation. (These graphs were called B_k-EPG graphs in [6].) In what follows, we will often identify the graph-theoretic concept (such as vertex) with the geometric object that represents it (the grid path).

3 k-bend EPG-graphs

In this section, we will show that the complete bipartite graph $K_{\ell,N}$ (for sufficiently large ℓ and N, depending on k) is not a k-bend EPG graph.

Theorem 1. *$K_{(k+3)/2,N}$ is not a k-bend EPG graph for N sufficiently large.*

Proof. Set $\ell = (k + 3)/2$ and $N = 2\ell^2(k + 1)^2 + 1 = \frac{1}{2}(k + 3)^2(k + 1)^2 + 1$. Let $V = A \cup B$ be the vertex partition of $K_{\ell,N}$, with $|B| = N$. Assume for contradiction that $K_{\ell,N}$ has a k-bend EPG representation, and let S be the grid-segments of the paths of vertices in A; S contains at most $\ell(k + 1)$ many segments.

Every path representing a vertex in B must intersect ℓ of the segments in S, and no two of these paths can intersect such segments at the same place since the vertices in B are an independent set.

Consider two segments s_1 and s_2 in S, and assume that P is a path that intersects both s_1 and s_2 and has at most one bend inbetween. If s_1 and s_2 have the same orientation (horizontal or vertical), then P cannot have a bend inbetween them, and hence must contain an endpoint of each of s_1 and s_2. If s_1 and s_2 have different orientation, then P must contain a bend between them, and it necessarily must be on the point where the two lines through s_1 and s_2

intersect. Either way, we can see that there can be at most two edge-disjoint paths that go from s_1 to s_2 with at most one bend inbetween.

Considering now all $\ell^2(k+1)^2$ possible pairs of grid segments, there are at most $2\ell^2(k+1)^2$ grid paths that can contain two of them consecutively with at most one bend inbetween. By choice of N, there is at least one vertex in B for which the representing path hence contains at least two bends between any two intersections with grid segments in S. Since the path intersects ℓ grid segments, it has at least $2\ell - 2 = k+1$ bends total, a contradiction.

We conjecture that the value of N in the above theorem is too large, and that already $K_{\ell,N}$, for some $\ell \in \theta(k)$ and $N \in \theta(k^2)$ is not a k-bend EPG graph.

Now we study when k-bend EPG representations actually exist.

Theorem 2. *Every bipartite graph* $(A \cup B, E)$ *with* $|A| = a$ *is a* $(2a-2)$*-bend EPG graph.*

Proof. We show first how to construct this for $K_{a,b}$. Our construction is essentially the same as the one by Golumbic et al. [6], except that one bend can be saved since the graph is bipartite. Represent each of the a vertices as a horizontal line, say at y-coordinates $1, 2, \ldots, a$. Define a "vertical zig-zag" to be the path $(1,1)-(2,1)-(2,2)-(1,2)-\ldots-(1,2i-1)-(2,2i-1)-(2,2i)-(1,2i)-\ldots$ that ends with horizontal segment $(1,b)-(2,b)$ or $(2,b)-(1,b)$. Represent each of the b vertices by such a zig-zag, translated horizontally as to make them disjoint. See Figure 1. One easily verifies that this is an EPG representation of $K_{a,b}$, and that the paths for the b vertices have $2a-2$ bends. For an arbitrary bipartite graph, we can obtain an EPG representation similarly, by omitting a "zig" or "zag" at any pair of vertices where no edge exists.

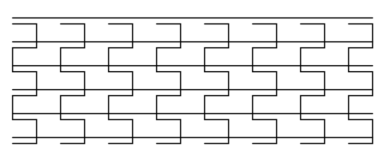

Fig. 1. The construction for the complete bipartite graph.

Notice in particular that $K_{(k+3)/2,N}$ is *not* a k-bend EPG graph for sufficiently large N, but it *is* a $(k+1)$-bend EPG graph by the previous theorem. So the bound on the number of bends for complete bipartite graphs is tight.

4 Planar graphs

A *planar graph* is a graph that can be drawn without crossing in the 2-dimensional plane. Much is known about planar graphs (see e.g. [9]). We will study here bounds on the number of bends needed in EPG representations, both for planar graphs and for some of their subclasses.

4.1 Planar graphs are 5-bend EPG-graphs

We show that every planar graph has a 5-bend EPG representation. Moreover, the layout of the paths for this is such that the paths only touch; they do not cross (in some sense, this is therefore a planar drawing.) We obtain this result by constructing a representation of the graph via touching T's using the canonical ordering for triangulated planar graphs. We explain the approach and these terms in detail in the following:

- First, fix an arbitrary planar drawing Γ of the graph. This fixes the planar embedding (the clockwise order of edges around each vertex), the faces (the maximal connected regions of $R^2 - \Gamma$) and the outer-face (the face which is unbounded.)
- Add vertices (and incident edges) to the graph to make it *triangulated*, i.e., it is planar and every face is a triangle. It is well-known that this can always be done using $O(n)$ new vertices. It suffices to show that the resulting graph is a 5-bend EPG graph; deleting the added vertices then gives a 5-bend EPG representation of the original graph.
- For triangulated planar graphs, there exists the *canonical ordering* introduced by de Fraysseix, Pach and Pollack [3]. This is a vertex order $v_1, \ldots, v_n$ such that $\{v_1, v_2, v_n\}$ is the outer-face, and for all $i \geq 3$, v_i is a vertex in the outer-face of the graph G_{i-1} induced by $v_1, \ldots, v_{i-1}$, and incident to a consecutive set of at least two vertices on the outer-face of G_{i-1}.
- The canonical ordering can be used to obtain many types of graph drawings for planar graphs, e.g., straight-line drawings, orthogonal drawings, visibility representations, and others (see e.g. [7]). We need here one special kind of drawing that to our knowledge has not been presented before, and may be of independent interest: Each vertex is represented by an upside-down T and two vertices are adjacent if and only if the T's touch. We call such a drawing a *T-drawing*.

  Lemma 1. *Every planar graph has a T-drawing, i.e., can be represented by touching Ts.*

  Proof. Assume the graph is triangulated and compute a canonical ordering. To create a T-drawing, we start with v_1 and v_2 as illustrated in Figure 2. While adding more vertices, we maintain the invariant that any vertex for which not all incident edges have been placed still has the upward ray from the center of the T unobstructed by any other vertex, and these unobstructed rays are ordered in the same way as the corresponding vertices on the outer-face of the current graph. Vertex v_i can then be added by placing it between the upward rays of its *leftmost predecessor* (i.e., the neighbour in G_{i-1} that comes first in clockwise order around the outer-face) and the *rightmost predecessor* defined similarly. All other neighbours w of v_i in G_{i-1} are not on the outer-face after adding v_i, and hence have no more incident edges; we let the ray from w end at the horizontal bar of v_i. See Figure 2. The upward ray for v_i can be placed anywhere, but for reasons that will be clear later, we will place it somewhere where there is no upward ray from below attaching at the bar.

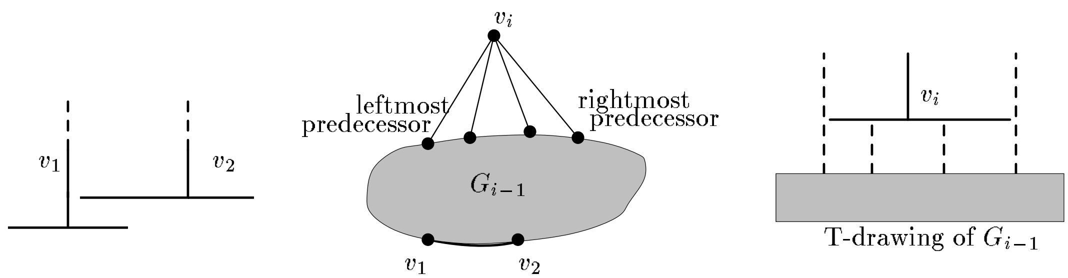

Fig. 2. A T-drawing of a planar triangulated graph obtained with the canonical ordering.

– Now we convert the T-drawing to 5-bend paths by tracing along the T. We need to add small handles at the ends of the T so that the corresponding paths intersect in edges, not just points. See Figure 3. Since no two edges attach at the same point of a vertex-segment, one easily verifies that this is an EPG representation of the graph. (We omitted the grid in the drawings, but one can create a grid by sorting the segments and assigning coordinates in this order.)

We conclude by stating the theorem that we proved:

Theorem 3. *Every planar graph is a 5-bend EPG graph.*

On the other hand, we can easily see that planar graphs are not 1-bend EPG graphs, by applying Theorem 1 to the planar graph $K_{2,N}$, for N sufficiently large. We strongly suspect that planar graphs are not 2-bend EPG graphs either (and possibly not even 4-bend EPG graphs), but this remains open.

4.2 Planar bipartite graphs

Now consider graphs that are both planar and bipartite. We have already seen in Theorem 1 that these are not always 1-bend EPG graphs, since $K_{2,N}$ is planar and bipartite.

Theorem 4. *Every planar bipartite graph is a 2-bend EPG.*

Proof. It is well-known [2] that every planar bipartite graph can be represented by touching horizontal and vertical line segments, i.e., for every vertex there exists a segment and (v, w) is an edge if and only if the corresponding segments touch. By thickening the segments, and then tracing their boundaries with a 2-bend path (a C for vertical and a U for horizontal), we obtain a 2-bend EPG representation.

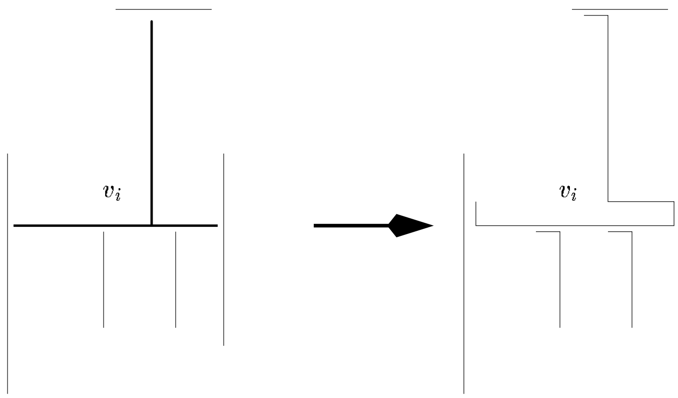

Fig. 3. Converting a T-drawing to a 5-bend EPG.

4.3 Outer-planar graphs

An *outer-planar graph* is a planar graph that can be drawn such that all vertices are on the outer-face. Since $K_{2,N}$ is not an outer-planar graph (for $N \geq 3$), one could suspect that these are all 1-bend EPG graphs. This is not the case.

Theorem 5. *Not all outer-planar graphs are 1-bend EPG-graphs.*

Proof. The graph G shown in Figure 4(a) is an outer-planar graph, but as we will argue now, it is not a 1-EPG graph.

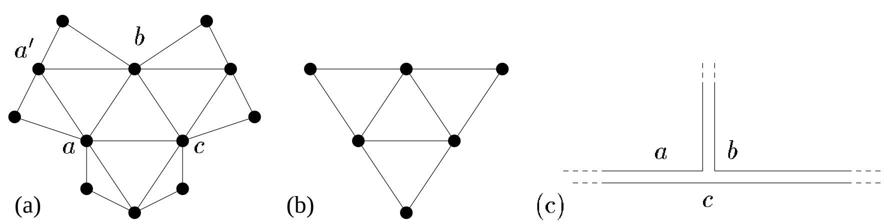

Fig. 4. (a) An outer-planar graph that is not a 1-bend EPG graph. (b) The 3-sun. (c) A claw-clique.

Consider a triangle in a 1-bend EPG representation. The union of three 1-bend paths cannot form a cycle in a rectangular grid, hence for each triangle the union of the paths forms a tree. It follows from the analysis of edge-intersection graphs of paths in trees [5] that these three paths either all intersect in one grid-edge (we say that they are an *edge-clique*), or they form a *claw-clique*, i.e., there exists a $K_{1,3}$ in the grid such that any degree-1 vertex of the claw is used by exactly two paths. See Figure 4(c).

Define a *3-sun* to be the graph that consists of a *central triangle*, and for every edge of the triangle a degree-2 vertex (called the *outrigger*) connected to the endpoints of the edge. From the work by Golumbic et al. [6], it follows that the central triangle of a 3-sun necessarily must be a claw-clique in any 1-bend EPG representation.

Let the central triangle of G be $\{a, b, c\}$. Since it is part of an induced 3-sun, it is a claw-clique. After possible renaming, assume that a and b have a bend in common, and let a' be the outrigger-vertex attached to edge (a, b). If $\{a, b, a'\}$ were a claw-clique, then the common bend of a and b would force the position of a', which hence would have to share grid-edges with c. But c and a' are not adjacent. So $\{a, b, a'\}$ are not a claw-clique. This is a contradiction, because $\{a, b, a'\}$ is also the central triangle of a 3-sun.

As for upper bounds, it is well-known that every outer-planar graph has a vertex ordering $v_1, \dots, v_n$ such that each vertex v_i has at most 2 neighbours in $v_1, \dots, v_{i-1}$. We say that it is *2-regular acyclic orientable.* We can give a 3-bend construction for any graph that has such an orientation (which includes more graphs, for example all series-parallel graphs, and even some non-planar graphs such as the graph obtained from K_n by subdividing all edges.)

Theorem 6. *Every 2-regular acyclic orientable graph is a 3-bend EPG graph.*

Proof. Let $v_1, \dots, v_n$ be a vertex order such that every vertex v_i has at most 2 earlier neighbours. We maintain the hypothesis that the graph induced by $v_1, \dots, v_i$ has a 3-bend EPG representation such that every vertex-path contains a horizontal grid-segment and a vertical grid-segment that does not intersect any other vertex-path; we call these the *free* segments. This is trivial for v_1; simply represent it by an arbitrary vertical and horizontal segment attached to each other.

To add vertex v_i to an existing drawing, find its earlier neighbours, say v_j and v_k. (The case of only one earlier neighbour is even easier.) Pick one grid edge e_j from the vertical free segment of v_j and one grid edge e_j from the horizontal free segment of v_k. Insert a new horizontal grid-line ℓ_j such that it splits e_j, and a new vertical grid-line ℓ_k such that it splits e_k.[3] Now define a path by using one part of e_j, going along ℓ_j until the crossing with ℓ_k, going along ℓ_k until e_k, and then using one part of e_k. See Figure 5.

One easily verifies that this path has three bends, intersects with the paths of v_j and v_k, and does not intersect any other paths. Moreover, v_j and v_k still have free segments (the other parts of e_j and e_k), and v_i also has free segments (along ℓ_j and ℓ_k), so the induction hypothesis holds with the new vertex.

We strongly suspect that this construction isn't optimal, especially for outer-planar graphs.

Conjecture 1. Every outer-planar graph is a 2-bend EPG graph.

[3] "Inserting" a grid-line really means that all coordinates of all endpoints and bends to the right/above are increased by 1.

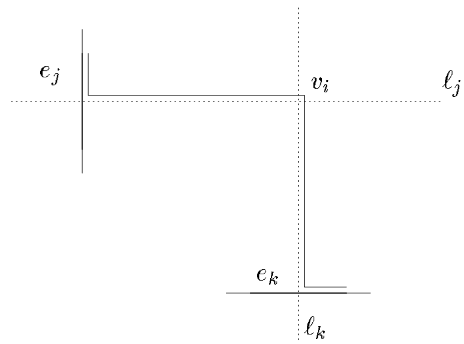

Fig. 5. The construction for 2-regular acyclic orientable graphs.

5 Remarks

We mention here, without proof, a few more results.

- A graph H is a line graph if there exists a graph G such that every vertex of H corresponds to an edge of G, and two vertices of H are adjacent if the two corresponding edges have an endpoint in common. It is easy to see that every line graph is a 2-bend EPG graph.
- A graph of pathwidth k has a vertex order $v_1, \dots, v_n$ such that for any i, at most k vertices in $v_1, \dots, v_i$ have a neighbour in $v_{i+1}, \dots, v_n$. We can show that graphs of pathwidth at most k are $(2k-2)$-bend EPG graphs.
- We say that a graph has a *κ-regular acyclic orientation* if it has an acyclic edge orientation with $indeg(v) \leq \kappa$ for all vertices. Then, as already observed by the authors, the construction in [6] yields that every κ-regular acyclic orientable graph has a 2κ-bend EPG graph.
  In Theorem 6, we improved this to 3 bends for $\kappa = 2$, and suspect that improvement is possible in general.

  Conjecture 2. Every κ-regular acyclic orientable graph is a $(2\kappa - 1)$-bend EPG graph.

  Graphs with a κ-regular acyclic orientation include graphs of arboricity at most κ (i.e., the edges can be split into κ forests). It is known that the arboricity of G is $\max_{H \subseteq G} \lceil |E(H)|/(|V(H)|-1) \rceil$ [8]. Graphs of arboricity κ includes planar graphs ($\kappa = 3$), planar bipartite graphs ($\kappa = 2$), graphs of treewidth κ, graphs of pathwidth κ, and many others. Of course, for some of these graphs we already found better constructions earlier.
- We say that a graph G has a *κ-regular orientation* if G has an edge orientation such that $indeg(v) \leq \kappa$ for all vertices v. It is known that the smallest κ for which G has a κ-regular orientation is $\max_{H \subseteq G} \lceil |E(H)|/|V(H)| \rceil$ [1]. We can show that every κ-regular orientable graph G is a $(2\kappa + 1)$-bend EPG graph.

We leave many open problems. An obvious one is to improve the upper or lower bounds for the number of bends for all graphs where this isn't tight yet. But more pressing are complexity issues. What is a recognition algorithm for 1-bend EPG graphs or k-bend EPG graphs? Is this NP-hard? What are time complexities of some problems in k-bend EPG graphs for small k? Since planar graphs are 5-bend EPG graphs, the 3-Coloring problem is NP-hard in 5-bend EPG graphs. Is it polynomial for smaller k?

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